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University of Alberta

NUMERICAL ANALYSIS OF AMERICAN OPTIONS

by

Hongtao Yang ©

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

in

Mathematical Finance

Department of Mathematical and Statistical Sciences
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UNIVERSITY OF ALBERTA

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Numerical Analysis of American Options** submitted by **Hongtao Yang** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematical Finance.

ABSTRACT

In this thesis we study numerical pricing of American options on stocks and zero-coupon bonds. For American options on stocks, based on new exact formulations of American option problems on bounded regions, we establish error estimates for finite element approximations of American option prices under admissible regularity. To the best of our knowledge, there were no earlier known results using this approach. Numerical results are also presented to examine our theoretical results and to compare them with other approaches by means of several examples, which show that our methods provide very rapid and accurate option prices, early exercise prices, and hedge ratios. For American put options on zero-coupon bonds under the CIR model, we show the existence and uniqueness of the weak solution by formulating the corresponding free boundary problems as parabolic variational inequalities. Since the degenerate term in the highest order derivative is removed, this formulation leads to a type of finite element methods which are numerically stable in a stronger sense. Besides, we also study finite volume methods for the original free boundary problem. Numerical examples show that our methods provide very accurate approximations of option prices. Stability and convergence are also obtained for the two methods. In addition, we give an error checking method which is a practical gauge of whether or not a numerical method converges and has achieved a good accuracy.

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Chapter 1

Introduction

In this chapter we introduce the basic concepts of option pricing in §1.1. The previous work on approximation methods for American options on stocks and bonds is reviewed in §1.2. Our work is summarized in §1.3.

1.1 Option pricing

The simplest financial option, a European call option, is a contract giving its owner the right (no obligation) to buy a prescribed asset for a prescribed price at a prescribed date in the future. The act of making this transaction is referred to as exercising the option. The prescribed asset is known as the underlying asset or, briefly, the underlying. The prescribed price is termed the exercise price or strike price, and the prescribed date is called the expiry date or maturity date. The individual who creates or issues a call option is termed the seller or writer, and the individual who buys a call option is termed the buyer or holder. In contrast to a European call option, a European put option gives its owner the right (no obligation) to sell the underlying asset. The counterparts of European options are American options which may be exercised at any time before or on the expiry date of the contracts. These standard options are colloquially called vanilla options, and others are called exotic options which are usually traded over-the-counter, for example,

binary options, compound options, chooser options, barrier options, Asian options, lookback options, and so on. An option can also be identified as path-dependent if its payoff at exercise depends, in some non-trivial way, on the past history of the underlying asset price as well as its price at exercise. Clearly, American options are path-dependent. Options are currently traded on stocks, stock indices, foreign currencies, futures contracts, and bonds.

Since the option confers on its holder a right with no obligation, it has some value which is referred to as the option price. One of the main concerns in option pricing theory is to find this price, in other words, how much one should pay for the right. There are two types of theoretical methods to evaluate an option: the equilibrium model method and the risk-neutral valuation method. The first method gives a partial differential equation which the option price satisfies. In the second method, the option price is expressed by an expectation with the aid of the martingale theory. Of course, the no-arbitrage argument is employed by both methods.

The theory of option pricing began in 1900 when Bachelier [11] deduced an option pricing formula based on the assumption that stock prices follow a standard Brownian motion. The main drawbacks of Bachelier's model are that the probability of negative stock prices is positive and the option prices may be greater than their respective stock prices. With the assumption that stock prices follow a geometric Brownian motion as first introduced to stock price change by Samuelson in 1960's [66], Black and Scholes [17] derived their famous formula of the European call option based on observable variables, which was published at the time roughly coincided with the opening of the Chicago Board of Trade. Since then, a flood of researches on option pricing has extended this model in various directions and sophisticated mathematical theories have been developed (see books [18, 32, 35, 41, 42, 52, 54, 62, 65] and references cited therein).

Most of the options that are traded on exchanges are of American type, e.g., American options on commodities or commodity futures, American call options on dividend paying stocks, American put options on dividend or non-

dividend paying stocks, American options on foreign currencies, American options on indices, American options on bonds, and so on. It is thus important to find appropriate ways to price American options. Since the holder of the American option can choose the exercise time, it is more complicated to analyze this situation than its European counterpart. For example, consider the American call option on a dividend paying stock. At any given time t, there exists a critical stock price $S^*(t)$ such that if the stock price S at time t is equal to or greater than $S^*(t)$ then the early exercise is optimal; otherwise, not optimal, that is, the option should be held. $S^*(t)$ is referred to as the early exercise price. It must be identified as a part of the solution of the modeling problem. This is the main difficulty associated with the valuation of American options in both theory and application. Unfortunately, no explicit closed-form analytic formulas have been found for American options with few exceptions. Therefore, approximation methods have to be used in practice. In this thesis, we shall study numerical methods for American options on stocks and zero-coupon bonds.

1.2 Previous Work

Pricing American options on stocks can be formulated as either optimal stopping problems or free boundary problems. The earlier theoretical work in this aspect includes the papers by McKean[59], Merton[60], van Moerbeke[69], Bensoussan[16], Karatzas[50, 51] and references cited therein. It is also well-known that the American option price equals the corresponding European option price plus an integral (called the early exercise premium). Since the integral involves the unknown early exercise price $S^*(t)$, this expression can not be used to evaluate the American option. But it can leads to integral equations for $S^*(t)$, which may be solved numerically ([40]). The theoretical work in this aspect includes the papers by Carr, Jarrow and Myneni[27], Jacka[46], Kim[53] and references cited therein.

In the last two decades, both analytic approximations and numerical meth-

ods for the valuation of American options have been investigated extensively. For analytic approximations, various procedures such as the interpolation method, the compound option approximation method, the quadratic approximation method, etc. (See [25, 49, 38, 15, 58] and references cited therein) have been introduced. Numerical methods can be developed along two lines, simulating the stochastic process of stock price and solving the free boundary problems. There are the binomial methods, the Monte Carlo methods, finite difference methods, and finite element methods (see [3, 24, 19, 21, 47, 57] and references cited therein). Except for quadratic approximation methods, approximate solutions obtained by other methods can be taken to the limit as the number of time/iteration steps goes to infinity. The convergence analysis is given for some methods, but to the best of our knowledge, there are no known error estimates for many of the methods.

Consider next the problem of pricing American options on bonds. Since bonds and bond options are interest rate derivatives, we shall rely on term structure models which feature the stochastic evolution of the short-term interest rate. The most popular models include the Vasiček model [70], the Brennan and Schwartz model [22], the Cox, Ingersoll, and Ross (CIR) model [31], etc. (see [54, 65] and references cited therein). For theoretical work on bonds and bond options we refer to [10, 18, 35, 65] and references cited therein. It should be pointed out that pricing American options on bonds can also be formulated as either optimal stopping time problems or free boundary problems (see [29]).

While there has been a large literature dealing with approximations of American options on stocks, pricing American bond options has not been paid much attention. Procedures such as finite difference methods, finite volume methods (also called box methods by other authors) and binomial methods have been used to evaluate bond options numerically. We refer the interested reader to [14, 43, 67, 68, 73] and references cited therein in this aspect. As pointed in [14] and [68], the finite difference method and the simplified binomial method are only convergent for certain combination of parameters. Although

the binomial method developed in [64] converges in all cases, its multiple jumps make the algorithm slower for large time steps. The numerical results in [14] indicate that the finite volume methods may converge in all cases.

1.3 Present Work

In this thesis we shall study finite element methods for pricing American options on stocks and finite element methods and finite volume methods for pricing American put options on zero-coupon bonds.

Recall that the classic Black-Scholes model for an American stock option leads to a free boundary problem with a degenerate partial differential operator (see [59, 60, 72]). To remove the degeneracy, a variable change is introduced which reduces the problem to a free boundary problem for the heat equation. Unfortunately, this problem needs to be solved over an infinite region in the space variable. In practice, this is resolved by solving the new problem over a large but finite region ([21, 47]). Two difficulties are thus introduced: (1) the simulations run over a "big" domain and thus are relatively slow; (2) an artificial boundary condition must be imposed, which effects the accuracy of the simulation. Accuracy problems particularly arise in situations where the interest rate is greater than the dividend, due to the specific nature of the convection term in the Black-Scholes' partial differential equation. This truncation approach was also employed in [57] where error estimates for a finite element method were obtained under suitable regularity conditions. In this paper we eliminate both difficulties by the introduction of a new nonlocal boundary condition, which is mathematically exact, and which allows us to reformulate the problem as a variational inequality on a very narrow region, without changing the solution.

To the best of our knowledge, there are relatively few papers in which error estimates are studied for parabolic inequalities (see [48], [71] and references cited therein). The main difficulty is the fact that the solutions of parabolic inequalities do not possess high regularity even if all the data are smooth

([23]). For an American option problem, the partial differential operator is degenerate and the initial condition and the obstacle (payoff) function are only in H^1. One can not expect that its solution has much global regularity. Recently, Badea and Wang ([12], [13]) have studied the weak solution by formulating the problem as a nonlinear variational problem. From their results, one may show that the solution of an American option problem is in $H^{2,1}$ locally. Under such regularity, motivated by the results of French and King [36] and Johnson [48], we shall construct a type of finite element methods for the associated variational inequalities which have been obtained and establish their error estimates. The existence, uniqueness and stability of finite element solutions are also obtained. Numerical examples are presented to demonstrate the theoretical results and to compare with other approaches.

For American put options on zero-coupon bonds under CIR model, we shall propose two types of numerical methods: finite volume methods and finite element methods. The finite volume methods considered in this thesis are different from those presented in [14] and [73] in two aspects: (1) more general discretization schemes in time are considered, e.g., the Crank-Nicholson scheme; (2) discrete linear complementarity problems are written in symmetric forms and thus can be solved exactly and rapidly by using the Brennan-Schwartz algorithm ([21]) instead of the projected SOR iteration process used in [14] and [73]. Our finite element methods are based on a new formulation of the original free boundary problem by introducing transforms. The degenerate factor in the highest order derivative term is removed. Thus the finite element methods are numerically stable in a stronger sense. The new formulation also allows us to show the existence and uniqueness of the weak solution. Stability and convergence analysis are established. Numerical examples show that all of our methods provide very accurate approximations of option prices.

It is very useful to provide a method to examine the convergence of a numerical method for a problem without an explicit closed-form solution. That is to say, to estimate the error in the approximate solution actually obtained. By using a similar procedure to that introduced in [2], we derive an equation

which is satisfied by the unknown bond put option price and early exercise interest rate (free boundary). By substituting the approximate answers in this equation and checking variations from zero, we obtain an error indicator. It gauges whether or not the numerical method converges and has achieved a good accuracy. Furthermore, it yields a practical means of determine how fine a grid should be employed in order to obtain answer with a desired accuracy.

To conclude the introduction, we give the outline of the thesis. In Chapter 2, we introduce spaces of functions and some useful results that will be used in the following chapters. In Chapter 3 and Chapter 4, we study American options on stocks and zero-coupon bonds, respectively. In Chapter 5, we summarize the results of the thesis and give future research directions.

Chapter 2

Spaces of Functions and Preliminary Results

In this chapter we introduce spaces of functions and some useful results that will be used in the following chapters.

2.1 Spaces of Functions

For a positive number X, let $\Omega = (0, X)$. For $p \geq 1$, we shall denote by $L^p(\Omega)$ the space of p-integrable functions on Ω and by $L^\infty(\Omega)$ the essentially bounded functions on Ω. For $v \in L^p(\Omega)$ $(1 \leq p \leq \infty)$, its norm will be denoted by $\|v\|_{L^p(\Omega)}$. The inner product on $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$.

For a nonnegative real number s we shall denote by $W^{s,p}(\Omega)$ the Sobolev space introduced in [1] and [56]. For $v \in W^{s,p}(\Omega)$, its norm will be denoted by $\|\cdot\|_{W^{s,p}(\Omega)}$. When s is a positive integer, we have

$$\|v\|_{W^{s,p}(\Omega)} = \left(\sum_{0 \leq j \leq s} \left\| \frac{d^j v}{dx^j} \right\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

In the case $p = 2$, we adopt the usual notation $H^s(\Omega)$ instead of $W^{m,p}(\Omega)$ and denote the norm by $\|\cdot\|_{s,\Omega}$.

Let $C_0^\infty(\Omega)$ be the set of all smooth functions on Ω with compact support in Ω. Denote by $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$. Also, $W_0^{s,2}(\Omega)$ is

denoted by $H_0^s(\Omega)$. As usual, the dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. For $\psi \in H^{-s}(\Omega)$, its norm is given by

$$\|\psi\|_{-s,\Omega} = \sup_{\phi \in H^s(\Omega)} \frac{\langle \psi, \phi \rangle}{\|\phi\|_{s,\Omega}},$$

where $\langle \cdot, \cdot \rangle$ denotes the dual product between $H^{-s}(\Omega)$ and $H^s(\Omega)$.

Lemma 2.1. *For any $v \in W_0^{1,p}(\Omega)(p > 1)$,*

$$\|x^{-1}v\|_{L^p(\Omega)} \leq \frac{p}{p-1}\|v_x\|_{L^p(\Omega)}, \tag{2.1}$$

where v_x denotes the derivative of v, and this inequality is optimal.

Proof. For $v \in W_0^{1,p}(\Omega)(p > 1)$, integration by parts gives

$$\|x^{-1}v\|_{L^p(\Omega)}^p = \frac{p}{p-1}\int_0^X (x^{-1}v)^{p-1} v_x dx.$$

Thus, inequality (2.1) follows from Hölder's inequality. For $\epsilon \in (0, 1)$, let $v^\epsilon(x) = x^{\epsilon + \frac{p-1}{p}}(X - x)$. Then $v^\epsilon \in W_0^{1,p}(\Omega)$ and by a simple calculation

$$\lim_{\epsilon \to 0+} \frac{\|x^{-1}v^\epsilon\|_{L^p(\Omega)}}{\|v_x^\epsilon\|_{L^p(\Omega)}} = \frac{p}{p-1}.$$

Thus inequality (2.1) is optimal. □

Remark 2.1. This lemma is the generalization of a results in [12] ($p = 2$).

In order to deal with boundary conditions at X, we also need the space $H_E^m(\Omega)$ which is the closure of all smooth functions v with $v(X) = 0$ in $H^m(\Omega)$.

Lemma 2.2. *For any $v \in H_E^1(\Omega)$,*

$$\frac{2}{2+X^2}\|v\|_{1,\Omega}^2 \leq \|v_x\|_{0,\Omega}^2 \leq \|v\|_{1,\Omega}^2. \tag{2.2}$$

Proof. The second inequality of (2.2) is obvious. For any $v \in H_E^1(\Omega)$, we have by Schwartz's inequality

$$v(x) = -\int_x^X v_x(y)dy \leq (X - x)^{\frac{1}{2}}\|v_x\|_{0,\Omega}.$$

Thus,

$$\|v\|_{0,\Omega}^2 \le \frac{X^2}{2}\|v_x\|_{0,\Omega}^2.$$

Hence,

$$\frac{2}{2+X^2}\|v\|_{1,\Omega}^2 \le \frac{2}{2+X^2}\left(\frac{X^2}{2}+1\right)\|v_x\|_{0,\Omega}^2 = \|v_x\|_{0,\Omega}^2.$$

This completes the proof of the lemma. □

Remark 2.2. Notice that $H_0^1(\Omega) \subset H_E^1(\Omega)$. Inequalities in (2.2) imply that $\|v_x\|_{0,\Omega}$ gives an equivalent norm to the usual norm $\|v\|_{1,\Omega}$ in $H_E^1(\Omega)$ and $H_0^1(\Omega)$. Henceforth, we shall use $\|v_x\|_{0,\Omega}$ as the norm in these spaces but still denote it by $\|v\|_{1,\Omega}$.

For a positive number T, let $J = (0,T)$. In the following we shall give an equivalent norm in $H^s(J)$ when $s \in \left[0,\frac{1}{2}\right)$. Recall that there is a positive constant C_1 such that (see Theorem 1.11.4 of [56])

$$\|\phi\|_{s,J} \le \|\varphi\|_{s,R} \le C_1\|\phi\|_{s,J}, \quad \forall\phi \in H^s(J), \tag{2.3}$$

where $R = (-\infty,+\infty)$, φ is the extension of ϕ by 0 outside J. Thus, for $s \in \left[0,\frac{1}{2}\right)$, the expression

$$|\phi|_{s,J} \equiv \|\varphi\|_{s,R} = \left(\int_R \left(1+\omega^2\right)^s |\widehat{\varphi}(\omega)|^2 d\omega\right)^{\frac{1}{2}} \tag{2.4}$$

defines an equivalent norm to $\|\phi\|_{s,J}$ in $H^s(J)$, where $\widehat{\varphi}$ is the Fourier transform of φ defined as follows:

$$\widehat{\varphi}(\omega) = \frac{1}{\sqrt{2\pi}}\int_R e^{-i\omega x}\varphi(x)dx.$$

From now on, we shall use $|\cdot|_{s,J}$ as the norm of $H^s(J)$, and shall use ϕ instead of φ if there is no confusion. For $\phi \in L^2(J)$, write

$$|\phi|_{-s,J} = \left(\int_R \left(1+\omega^2\right)^{-s} |\widehat{\phi}(\omega)|^2 d\omega\right)^{\frac{1}{2}}.$$

It will be shown that $|\phi|_{-s,J}$ gives an equivalent norm to $\|\phi\|_{-s,J}$ in $H^{-s}(J)$.

Lemma 2.3. *For* $s \in (0, \frac{1}{2})$,

$$|||\phi|||_{s,J} = \left(\int_R |\omega|^{2s} |\widehat{\varphi}(\omega)|^2 d\omega \right)^{\frac{1}{2}}$$

gives another equivalent norm in $H^s(J)$.

Proof. Since

$$2^{-s}\left(1+\omega^2\right)^s \leq 1 + |\omega|^{2s} \leq 2\left(1+\omega^2\right)^s, \tag{2.5}$$

it suffices to show that there is a positive constant C_2 such that

$$\|\phi\|_{0,J} \leq C_2 |||\phi|||_{s,J}, \ \forall \phi \in H^s(J). \tag{2.6}$$

Suppose that (2.6) is not true. Then there is a sequence of functions $\{\phi_n\}$ in $H^s(J)$ such that

$$\|\phi_n\|_{0,J} \geq n |||\phi_n|||_{s,J}, \quad |\phi_n|_{s,J} = 1, \quad n = 1, 2, \ldots$$

Since the embedding from $H^s(J)$ to $L^2(J)$ is compact, we may assume that $\{\phi_n\}$ converges to ϕ in $L^2(J)$. Thus, by (2.5),

$$\lim_{n \to \infty} |||\phi_n|||_{s,J} = 0, \tag{2.7}$$

$$\lim_{n \to \infty} \|\phi_n\|_{0,J} \geq 2^{-s}. \tag{2.8}$$

Equation (2.7) implies that $\phi = 0$ a.e. on J. Hence, by (2.8),

$$0 = \|\phi\|_{0,J} = \lim_{n \to \infty} \|\phi_n\|_{0,J} \geq 2^{-s},$$

which is a contradiction. Therefore, (2.3) holds. □

To conclude this section, we briefly introduce spaces of functions related to time variable. The detailed definitions of these spaces can be found in [56]. To this end, we consider a B-valued function $v : t \in [0,T] \to v(t) \in B$, where B will be one of $H^s(\Omega)$, $H_0^s(\Omega)$ or $H_E^s(\Omega)$. Then we can define $W^{m,p}(0,T;B)$ similar to $W^{m,p}(\Omega)$ and denote $W^{0,p}(0,T;B)$ by $L^p(0,T;B)$, where m is a nonnegative number. The space $H^{2m,m}(Q)$ is defined to be

$L^2(J; H^{2m}(\Omega)) \cap H^m(J; L^2(\Omega))$ and its norm is denoted by $\|\cdot\|_{2m,m}$, where $Q = \Omega \times J$. When m is a nonnegative integer, we have for $v \in H^{2m,m}(Q)$

$$\|v\|_{2m,m}^2 = \int_0^T \|v(t)\|_{2m,D}^2 dt + \int_0^T \sum_{j=0}^m \left\| \frac{d^j v}{dt^j}(t) \right\|_{0,\Omega}^2 dt.$$

From Proposition 4.2.1 and Theorem 4.2.1 of [56], we have the following useful interpolation inequality and trace estimate:

$$\|v\|_{1,\frac{1}{2}}^2 \le C_4 \|v\|_{2,1} \|v\|_{0,0}, \ \forall v \in H^{2,1}(Q), \tag{2.9}$$

$$\|v(0,\cdot)\|_{m-\frac{1}{4},J} \le C_5 \|v\|_{2m,m}, \ \forall v \in H^{2m,m}(Q), \ m = \tfrac{1}{2}, \ 1. \tag{2.10}$$

2.2 Two Integral Operators

Consider the integral operator B defined by

$$B\phi(t) = \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-\frac{1}{2}} \phi(s) ds. \tag{2.11}$$

For $\psi(t) \in C_0^\infty(J)$, solving the following equation

$$B\phi(t) = \psi(t), \ t > 0,$$

we obtain (see Chapter 8 of [26]),

$$\phi(t) = A\psi(t), \ t > 0$$

where

$$A\psi(0,t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \left(\int_0^t (t-s)^{-\frac{1}{2}} \psi(0,s) ds \right). \tag{2.12}$$

Since

$$\int_0^t (t-s)^{-\frac{1}{2}} \psi(s) ds = \int_0^t s^{-\frac{1}{2}} \psi(t-s) ds,$$

we have

$$A\psi(t) = \frac{dB\psi(t)}{dt} = B\frac{d\psi(t)}{dt}, \ t > 0 \tag{2.13}$$

Thus,

$$AB\psi(t) = B\frac{dB\psi(t)}{dt} = BA\psi(t) = \psi(t).$$

Hence, A and B are inverse to each other in $C_0^\infty(J)$.

Lemma 2.4. (i) *A is an isomorphism from $H^{\frac{1}{4}}(J)$ to $H^{-\frac{1}{4}}(J)$ and there is a positive constant C_3 such that*

$$\langle A\phi, \phi\rangle \geq C_3\|\phi\|^2_{\frac{1}{4},J}, \quad \forall\phi \in H^{\frac{1}{4}}(J). \tag{2.14}$$

(ii) *B is an isomorphism from $H^{-\frac{1}{4}}(J)$ to $H^{\frac{1}{4}}(J)$ and*

$$\langle B\phi, \phi\rangle \geq \frac{1}{\sqrt{2}}|\phi|^2_{-\frac{1}{4},J}, \ \forall\phi \in L^2(J). \tag{2.15}$$

Proof. (i) Recalling the Lax-Milgram Lemma (Theorem 1.1.3 in [30]), we only need to show that A is a bounded linear operator from $H^{\frac{1}{4}}(J)$ to $H^{-\frac{1}{4}}(J)$ and (2.14) holds. For $\phi \in C_0^\infty(J)$, we can extend $A\phi(t)$ from $t \in J$ to $t \in R$ naturally and then have

$$A\phi(t) = \int_R K(t-s)\phi(s)ds \ \text{ in } R,$$

where $K(t) = -\frac{1}{2\sqrt{\pi}}t_+^{-\frac{3}{2}}$ and the integral is understood in the distribution sense. Since the Fourier transform of $K(t)$ (see Chapter II in [37]) is

$$\widehat{K}(\omega) = \frac{1}{\sqrt{4\pi}}|\omega|^{\frac{1}{2}}[1 + i(H(\omega) - H(-\omega))],$$

where $H(\omega)$ is the Heaviside function. We have for $\psi \in C_0^\infty(J)$,

$$\langle A\phi, \psi\rangle = \int_R A\phi(t)\psi(t)dt = \int_R \widehat{A\phi}(t)\overline{\widehat{\psi}}(t)dt = \int_R \sqrt{2\pi}\widehat{K}(\omega)\widehat{\phi}(\omega)\overline{\widehat{\psi}}(\omega)d\omega.$$

Hence,

$$\begin{aligned}
|\langle A\phi, \psi\rangle| &\leq \int_R |\omega|^{\frac{1}{2}} \left|\widehat{\phi}(\omega)\right| \left|\widehat{\psi}(\omega)\right| d\omega \\
&\leq \int_R (1+\omega^2)^{\frac{1}{8}} \left|\widehat{\phi}(\omega)\right| (1+\omega^2)^{\frac{1}{8}} \left|\widehat{\psi}(\omega)\right| d\omega \\
&\leq \|\phi\|_{\frac{1}{4},J}\|\psi\|_{\frac{1}{4},J}.
\end{aligned}$$

Since $C_0^\infty(J)$ is dense in $H^{\frac{1}{4}}(J)$ (see Theorem 1.11.1 of [56]), the above inequalities imply that A is a bounded linear operator from $H^{\frac{1}{4}}(J)$ to $H^{-\frac{1}{4}}(J)$. Notice that

$$\langle A\phi, \phi\rangle = \Re\langle A\phi, \phi\rangle = \frac{1}{\sqrt{2}}\int_R |\omega|^{\frac{1}{2}} \left|\widehat{\phi}(\omega)\right|^2 d\omega,$$

where $\Re$ means the real part of a complex number. Hence (2.14) follows from Lemma 2.3.

(ii) From (3.24) and (3.25), we obtain that

$$B(A\phi) = A(B\phi) = \phi, \quad \forall \phi \in C_0^\infty(J).$$

Thus B is the inverse of A and then it is an isomorphism from $H^{-\frac{1}{4}}(J)$ to $H^{\frac{1}{4}}(J)$. For $\phi \in C_0^\infty(J)$, we have

$$\widehat{B\phi}(\omega) = \frac{1}{\sqrt{2}}|\omega|^{-\frac{1}{2}}[1 + i(H(-\omega) - H(\omega))]\widehat{\phi}(\omega).$$

Then,

$$\begin{aligned}\langle B\phi, \phi\rangle = \Re\langle B\phi, \phi\rangle &= \frac{1}{\sqrt{2}}\int_R |\omega|^{-\frac{1}{2}}\left|\widehat{\phi}(\omega)\right|^2 d\omega \\ &\geq \frac{1}{\sqrt{2}}\int_R \left(1+\omega^2\right)^{-\frac{1}{4}}\left|\widehat{\phi}(\omega)\right|^2 d\omega = \frac{1}{\sqrt{2}}|\phi|^2_{-\frac{1}{4},J}.\end{aligned}$$

Since $C_0^\infty(J)$ is also dense in $L^2(J)$, (2.15) is true. □

Remark 2.3. For $\alpha \in (0, 1/2)$, let

$$B_\alpha\phi(t) = \frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\phi(s)ds.$$

It is shown in [33] that

$$\langle B_\alpha\phi, \phi\rangle \geq \cos(\pi\alpha)|\phi|^2_{-\alpha,J}, \ \forall \phi \in L^2(J). \tag{2.16}$$

It follows from Theorem 8.1.1 of [26] that the inverse of B_α is

$$A_\alpha\phi(t) = \frac{1}{\Gamma(1-2\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-2\alpha}\phi(s)ds$$

By our method, one can also prove that A_α is an isomorphism from $H^\alpha(J)$ to $H^{-\alpha}(J)$ and there is a positive constant C_α such that

$$\langle A_\alpha\phi, \phi\rangle \geq C_\alpha\|\phi\|^2_{\alpha,J}, \ \forall \phi \in H^\alpha(J).$$

Thus, B_α also is an isomorphism from $H^{-\alpha}(J)$ to $H^\alpha(J)$. Then it follows from (2.16) that for some positive constant D_α,

$$|\psi|_{-\alpha,J} \leq D_\alpha\|\psi\|_{-\alpha,J}, \ \forall \psi \in L^2(J).$$

By observing that for any $\phi \in H^\alpha(J),\ \psi \in L^2(J)$

$$\langle \psi, \phi \rangle \leq \|\phi\|_{\alpha,J} |\psi|_{-\alpha,J}, \tag{2.17}$$

we have

$$\|\psi\|_{-\alpha,J} \leq |\psi|_{-\alpha,J}, \quad \forall \psi \in L^2(J).$$

Therefore, $\|\cdot\|_{-\alpha,J}$ and $|\cdot|_{-\alpha,J}$ are equivalent norms in $H^{-\alpha}(J)$.

Remark 2.4. For $\psi \in H^1(J)$ with $\psi(0) = 0$, it follows from (2.13) that

$$A\psi(t) = B\psi'(t). \tag{2.18}$$

This identity will be used in computing $A\psi(t)$ in our finite element methods for American options on stocks in the next chapter.

Chapter 3

Finite Element Methods for American Options on Stocks

In this chapter, we consider finite element methods for American option problems. American option problems on stocks are introduced in §3.1. In §3.2 The free boundary problems for both American put options and American call options are exactly reformulated as variational inequalities of the heat equation with nonlocal boundary conditions on bounded domains. As a result of our formulations, we give a new proof of the put-call symmetry relations. In §3.3 approximations of the variational inequalities are given by using piecewise linear finite elements and the general time discrete scheme. We shall prove the existence, uniqueness and stability of approximate solutions. Error estimates of order $O\left(k^{\frac{1}{2}} + h\right)$ are established in §3.4. In the final section, §3.5, numerical experiments are explicitly presented to examine the convergence of our approach as parameters are varied. We also present actual applications to specific examples and detailed comparison with other approaches, such as the binomial method, finite difference method, integral equation method, quadratic approximation, and compound option approximation. Numerical results show that for all types of parameter combinations our approach provides very rapid accurate option prices, hedge ratios and early exercise prices.

3.1 American Options on Stocks

Consider an American option (call or put) on a stock paying a continuous dividend d with strike price $\$K$ and expiry date T_0, where d is non-negative constant. As usual, we assume that the capital market is frictionless and arbitrage-free with continuous trading possibilities and that stock price process $S(t)$ follows a lognormal diffusion process under the objective probability P:

$$\frac{dS(t)}{S(t)} = (\mu - d)dt + \sigma dB_t,$$

where the return volatility σ and expected return μ are assumed constants, B_t is a standard Brownian motion under P, and today's time is $t = 0$. We also assume that the risk-less interest rate r is a non-negative constant. It has been shown that there is a probability measure Q such that the discounted stock price $e^{-(r-d)t}S(t)$ is a martingale with respect to Q (called the martingale measure). Indeed, $S(t)$ satisfies the following dynamics under Q:

$$\frac{dS(t)}{S(t)} = (r - d)dt + \sigma dW_t,$$

where W_t is a standard Brownian motion under Q. It follows from Ito's lemma that

$$S(t) = S_0 e^{\sigma W_t + (r-d-\sigma^2/2)t}.$$

It has been shown that the fair price of the American option at time t, $V(S(t), t; K, \sigma, r, d, T_0)$ is given by (see [50, 63])

$$V(S(t), t; K, \sigma, r, d, T_0) = \operatorname*{ess\ sup}_{\tau \in \mathcal{T}_{[t,T_0]}} E^Q \left[e^{-r(\tau-t)} \Phi(S_\tau) \middle| \mathcal{F}_t \right], \tag{3.1}$$

where E^Q is the expectation under Q, $\{\mathcal{F}_t\}$ is the filtration generated by W_t, $\mathcal{T}_{[t,T_0]}$ is the set of all stopping time taking value in $[t, T_0]$, $\Phi(S)$ is the payoff defined as

$$\Phi(S) = \begin{cases} (S-K)_+ & \text{for a call,} \\ (K-S)_+ & \text{for a put,} \end{cases}$$

$z_+ = \max(z, 0)$, and

$$S_\tau = S(t)e^{(r-d-\sigma^2/2)(\tau-t)+\sigma(W_\tau - W_t)}. \tag{3.2}$$

In addition, the early exercise price is the smallest value of S for a call or the largest value of S for a put such that

$$V(S,t;K,\sigma,r,d,T_0)=\Phi(S).$$

From now on we shall use $c(S,t;K,\sigma,r,d,T_0)$ and $p(S,t;K,\sigma,r,d,T_0)$ for the option price and $S_c^*(t;K,\sigma,r,d,T_0)$ and $S_p^*(t;K,\sigma,r,d,T_0)$ for the early exercise price for a call and a put instead of $V(S,t;K,\sigma,r,d,T_0)$ and $S^*(t;K,\sigma,r,d,T_0)$, respectively. If no confusion arises, we shall omit all or part of the parameters K, σ, r, d, T_0 in these notations.

It is well known that the pair $c(S,t)$ and $S_c^*(t)$ is the solution of the following free boundary value problem ([59], [60]):

$$c_t+\frac{1}{2}\sigma^2S^2c_{SS}+(r-d)Sc_S-rc=0,\quad 0<S<S_c^*(t),\ 0<t\le T_0, \tag{3.3}$$

$$c(S,t)>(S-K)_+,\quad 0<S<S_c^*(t),\ 0<t\le T_0, \tag{3.4}$$

$$c(S,t)=(S-K)_+,\quad S\ge S_c^*(t),\ 0\le t\le T_0, \tag{3.5}$$

$$c(S_c^*(t),t)=(S_c^*(t)-K)_+,\ c_S(S_c^*(t),t)=1,\quad 0<t\le T_0, \tag{3.6}$$

$$c(S,T_0)=(S-K)_+,\quad S\ge 0, \tag{3.7}$$

$$c(0,t)=0.\ 0\le t\le T_0. \tag{3.8}$$

And the pair $p(S,t)$ and $S_p^*(t)$ is the solution of the following free boundary value problem ([46]:

$$p_t+\frac{1}{2}\sigma^2S^2p_{SS}+(r-d)Sp_S-rp=0,\quad S>S_p^*(t),\ 0<t\le T_0, \tag{3.9}$$

$$p(S,t)>(K-S)_+,\quad S>S_p^*(t),\ 0<t\le T_0, \tag{3.10}$$

$$p(S,t)=(K-S)_+,\quad 0\le S\le S_p^*(t),\ 0\le t\le T_0, \tag{3.11}$$

$$p(S_p^*(t),t)=(K-S_p^*(t))_+,\ p_S(S_p^*(t),t)=-1,\quad 0<t\le T_0, \tag{3.12}$$

$$p(S,T_0)=(K-S)_+,\quad S\ge 0, \tag{3.13}$$

$$\lim_{S\to+\infty}p(S,t)=0.\ 0\le t\le T_0. \tag{3.14}$$

Remark 3.1. We may assume that $d>0$ for a call. In fact, if $d=0$, then the value of the American call option equals the value of the corresponding

European call option (see [60]). By the same reason, we assume that $r > 0$ for a put (see [54]).

Remark 3.2. It is easy to check from (3.1) and (3.2) that

$$V(S,t;K_1,\sigma,r,d,T_0) = \frac{K_1}{K_2} V(K_2 S/K_1, t; K_2, \sigma, r, d, T_0)$$

for the options with strike prices K_1 and K_2. In particular, we have

$$V(S,t;K,\sigma,r,d,T_0) = KV(S/K, t; 1, \sigma, r, d, T_0).$$

Thus, the option price and the "Greeks" can be evaluated by considering the option with the strike price \$1. In particular, we have

$$V_S(S,t;K,\sigma,r,d,T_0) = V_S(S/K, t; 1, \sigma, r, d, T_0),$$

Remark 3.3. It is well-known ([54], [60]) that $S_p^*(t)$ is a non-increasing function and

$$S_0^c \le S_c^*(t) \le S_\infty^c, \quad 0 \le t \le T_0, \tag{3.15}$$

where

$$S_0^c = S_c^*(T_0) = \max\left(\frac{rK}{d}, K\right), \quad S_\infty^c = \frac{\sqrt{\beta} - \alpha}{\sqrt{\beta} - \alpha - 1} K,$$
$$\alpha = \frac{r-d}{\sigma^2} - \frac{1}{2}, \quad \beta = \alpha^2 + \frac{2r}{\sigma^2}.$$

Also, $S_c^*(t)$ is a non-decreasing function and

$$S_\infty^p \le S_p^*(t) \le S_0^p, \quad 0 \le t \le T_0. \tag{3.16}$$

where

$$S_0^p = S_p^*(T_0) = \min\left(\frac{rK}{d}, K\right), \quad S_\infty^p = \frac{\alpha + \sqrt{\beta}}{1 + \alpha + \sqrt{\beta}} K.$$

3.2 Reformulation

In this section we reduce the original free boundary problems into free boundary problems on finite domains by introducing exact artificial boundary conditions.

3.2.1 Call

Let $T = \frac{1}{2}\sigma^2 T_0$. By using the standard transforms:

$$c(S,t) = Ke^{-\alpha x - \beta\tau}\phi(x,\tau), \ \ T_0 - t = \frac{2\tau}{\sigma^2}, \ S = Ke^x, \tag{3.17}$$

(3.3)–(3.8) become

$$\phi_\tau - \phi_{xx} = 0, \ \phi(x,\tau) > g^c(x,\tau), \ \ x < x_c^*(\tau), \ 0 < \tau \leq T, \tag{3.18}$$

$$\phi(x,\tau) = g^c(x,\tau), \ \ x \geq x_c^*(\tau), \ 0 \leq \tau \leq T, \tag{3.19}$$

$$\phi(x_c^*(\tau),\tau) = g^c(x_c^*(\tau),\tau), \ \phi_x(x_c^*(\tau),\tau) = g_x^c(x_c^*(\tau),\tau), \ \ 0 < \tau \leq T, \tag{3.20}$$

$$\phi(x,0) = g^c(x,0), \ \ -\infty < x < \infty, \tag{3.21}$$

$$\lim_{x \to -\infty} e^{-\alpha x - \beta\tau}\phi(x,\tau) = 0, \ \ 0 \leq \tau \leq T, \tag{3.22}$$

where

$$g^c(x,\tau) = e^{\alpha x + \beta\tau}(e^x - 1)_+, \ \ x_c^*(\tau) = \ln(S_c^*(T_0 - 2\tau/\sigma^2)/K).$$

It follows from 3.15 that

$$0 \leq x_c^*(t) \leq X_c, \ \ t \in [0,T].$$

where $X_c = \ln(S_\infty^c/K)$. We shall see in §6 that X_c is very small.

In order to solve (3.18)–(3.22) on $[0,X_c] \times [0,T]$, we must provide a boundary condition at $x = 0$. From (3.18), (3.21) and (3.22), we have by Theorem 19.3.3 of [26]

$$\phi(x,t) = 2\int_0^t G(x,t-s)\phi_x(0,s)ds, \ \ 0 \leq t \leq T, \ x < 0, \tag{3.23}$$

where

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

is the fundamental solution of the heat equation. Letting $x \to 0^-$ in (3.23), we get

$$\phi(0,t) = B\phi_x(0,t), \ \ 0 \leq t \leq T, \tag{3.24}$$

where B is defined as in(2.11). Recalling (2.13), we obtain

$$\phi_x(0,t) = A\phi(0,t), \quad 0 \le t \le T, \tag{3.25}$$

where A is defined as in (2.12). Equation (3.25) is the desired exact boundary condition at $x = 0$. This is a complicated and nonlocal condition, but it means that we can restrict our consideration to $x \ge 0$.

By letting $\phi = u + g^c$, we can reformulate (3.18)–(3.22) into

$$u_t - u_{xx} = f^c, \quad u(x,t) > 0, \ 0 < x < x_c^*(t), \ 0 < t \le T, \tag{3.26}$$

$$u(x,t) = 0, \quad x_c^*(t) \le x \le X_c, \ 0 \le t \le T, \tag{3.27}$$

$$u(x_c^*(t),t) = 0, \ u_x(x_c^*(t),t) = 0, \quad 0 < t \le T, \tag{3.28}$$

$$u_x(0,t) = Au(0,t) - b(t), \quad 0 \le t \le T, \tag{3.29}$$

$$u(x,0) = 0, \quad 0 \le x \le X_c, \tag{3.30}$$

where

$$f^c(x,t) = \frac{2}{\sigma^2}\left(r - de^x\right)e^{\alpha x + \beta t}, \quad b(t) = e^{\beta t}.$$

Notice that

$$f^c(x,t) < 0, \quad x_c^*(t) \le x \le X_c, \ 0 \le t \le T.$$

The above problem is equivalent to the following variational inequality:

$$u_t - u_{xx} \ge f^c, \ u \ge 0, \ 0 < x < X_c, \ 0 < t \le T, \tag{3.31}$$

$$\left(u_t - u_{xx} - f^c\right)u = 0, \ 0 < x < X_c, \ 0 < t < T, \tag{3.32}$$

$$u(x,0) = 0, \ 0 \le x \le X_c, \tag{3.33}$$

$$u_x(0,t) = Au(0,t) - b(t), \ 0 \le t \le T, \tag{3.34}$$

$$u(X_c,t) = 0, \ 0 \le t \le T. \tag{3.35}$$

Remark 3.4. By Theorem 19.3.2 of [26], we also have

$$\phi(x,t) = -\frac{x}{\sqrt{4\pi}}\int_0^t (t-s)^{-\frac{3}{2}} e^{-\frac{x^2}{4(t-s)}}\phi(0,s)ds, \quad x < 0.$$

Hence, it follows from some simple calculations that

$$c(S,t) = \begin{cases} S - K, & S > S_\infty^c, \\ Ke^{-\alpha x - \beta\tau}u(x,\tau) + S - K, & K \le S \le S_\infty^c, \\ -\displaystyle\int_0^\tau E(x,\tau,s)u(0,s)ds, & 0 < S < K, \end{cases} \tag{3.36}$$

for $t \in [0, T_0]$, where $x = \ln(S/K)$, $\tau = \frac{\sigma^2}{2}(T_0 - t)$,

$$E(x, \tau, s) = \frac{Kx}{\sqrt{4\pi}}(\tau - s)^{-\frac{3}{2}} e^{-\frac{x^2}{4(\tau - s)} - \alpha x - \beta \tau},$$

and that $\phi(0, t) = u(0, t)$ is used.

3.2.2 Put

Now we consider an American put option. By the same transforms of t and S as in (3.17) and

$$p(S, t) = Ke^{-\alpha x - \beta\tau}\psi(x, \tau),$$

(3.9)–(3.14) becomes as

(3.37) $\psi_\tau - \psi_{xx} = 0, \ \psi(x, \tau) > g^p(x, \tau), \quad x > x_p^*(\tau), \ 0 < \tau \leq T,$

(3.38) $\psi(x, \tau) = g^p(x, \tau), \quad x \leq x_p^*(\tau), \ 0 \leq \tau \leq T,$

(3.39) $\psi(x_p^*(\tau), \tau) = g^p(x_p^*(\tau), \tau), \ \psi_x(x_p^*(\tau), \tau) = g_x^p(x_p^*(\tau), \tau), \quad 0 < \tau \leq T,$

(3.40) $\psi(x, 0) = g^p(x, 0), \quad -\infty < x < +\infty,$

(3.41) $\lim\limits_{x \to +\infty} e^{-\alpha x - \beta\tau}\psi(x, \tau) = 0, \quad 0 \leq \tau \leq T,$

where

$$g^p(x, \tau) = e^{\alpha x + \beta\tau}(1 - e^x)_+, \quad x_p^*(\tau) = \ln(S_p^*(T_0 - 2\tau/\sigma^2)/K).$$

By the inequalities in 3.16, we have

$$-X_p \leq x_p^*(t) \leq 0, \quad t \in [0, T],$$

where $X_p = \ln(K/S_\infty^p)$.

Setting $\psi(x, t) = v(-x, t) + g^p(x, t)$, yields the following variational inequality:

(3.42) $\qquad v_t - v_{xx} \geq f^p, \quad v \geq 0, \quad 0 < x < X_p, \ 0 < t \leq T,$

(3.43) $\qquad (v_t - v_{xx} - f^p)\, v = 0, \quad 0 < x < X_p, \ 0 < t < T,$

(3.44) $\qquad v(x, 0) = 0, \quad 0 \leq x \leq X_p,$

(3.45) $\qquad v_x(0, t) = Av(0, t) - b(t), \quad 0 \leq t \leq T,$

(3.46) $\qquad v(X_p, t) = 0, \quad 0 \leq t \leq T,$

where $f^p(x,t) = -f^c(-x,t)$ and $b(t)$ is the same as before.

Remark 3.5. As for the call price $c(S,t)$ (see Remark 3.4), $p(S,t)$ can be expressed by using $v(x,t)$ as follows:

$$p(S,t) = \begin{cases} K - S, & 0 \le S < S^p_\infty, \\ Ke^{-\alpha x - \beta\tau} v(-x,\tau) + K - S, & S^p_\infty \le S \le K, \\ \displaystyle\int_0^\tau E(x,\tau,s)v(0,s)ds, & S > K, \end{cases}$$

for $t \in [0, T_0]$, where $x = \ln(S/K)$, $\tau = \frac{\sigma^2}{2}(T_0 - t)$.

Remark 3.6. We consider the call and put options with the same strike price K on the same stock. It is easy to check the following identities:

$$X_c(r,d) = X_p(d,r), \quad b(t;r,d) = b(t;d,r),$$
$$f^p(x,t;d,r) = f^c(x,t;r,d), \quad g^c(x,t;r,d) = g^p(-x,t;d,r).$$

Hence (3.31)–(3.35) and (3.42)–(3.46) are the same problem after r and d are switched for the put option. Thus,

$$x^*_p(t;d,r) = -x^*_c(t;r,d), \quad t \in [0,T], \tag{3.47}$$

$$\psi(x,t;d,r) = \phi(-x,t;r,d), \quad t \in [0,T], \ x \in [0, X_p(d,r)]. \tag{3.48}$$

Furthermore, from (3.18)–(3.22) and (3.37)–(3.41), we can show that (3.48) holds for all $x \in (-\infty, +\infty)$. Equations (3.47)–(3.48) imply the following put-call symmetry relations:

$$S^*_c(t;r,d)S^*_p(t;d,r) = K^2, \quad c(S,t;r,d) = Sp(K^2/S,t;d,r)/K,$$

which have been proved by other methods (see Chapter 4 of [54]). From the above discussion, both call and put option prices can be computed by employing the same software codes used for solving (3.31)–(3.35).

Remark 3.7. One can also use (3.3)–(3.8) and (3.9)–(3.14) to evaluate American options on commodities and commodity futures. To this end, just take $d = r - b$, where b is the cost of carrying the commodity. It is reasonable to assume that $r > b$ for call options and thus $d > 0$. For the detailed discussion in this aspect, we refer the interested reader to [15].

Remark 3.8. As mentioned earlier, the existence of weak solution to problems (3.3)–(3.8) and (3.9)–(3.14) have been obtained in [12] and [13], to which we refer the interested readers. From the results in these two papers , one may show that $u \in H^{2,1}((0, X_c) \times (0,T))$ and $v \in H^{2,1}((0, X_p) \times (0,T))$.

3.3 Finite Element Solutions

In this section and in the next section, we shall introduce a type finite of element methods for the variational inequalities derived in Section 3.2 and establish their error estimates. Recalling Remark 3.6, we shall only deal with American call options from now on. For simplicity, we omit reference to c in f^c and X_c.

Let

$$\begin{aligned}\Xi = \{v \in L^2(J; H^1(\Omega)) &: v(0,t) \in H^{\frac{1}{4}}(J),\\ &v(X,t) = 0 \text{ a.e. on } J, \text{ and } v(x,t) \geq 0 \text{ a.e. on } Q\},\end{aligned}$$

where $\Omega = (0, X)$, $J = (0,T)$ and $Q = \Omega \times J$. Define the bilinear form

$$a(u,v) = \int_\Omega u_x v_x dx.$$

The variational form of (3.31)–(3.35) is: Find $u \in \Xi$ such that $u_t \in L^2(J; L^2(\Omega))$, $u(x,0) = 0$ and

$$\begin{aligned}\int_J ((u_t, u-v) + a(u, u-v))dt &+ \langle Au, u-v\rangle\\ &\leq \int_J (f, u-v)dt + \langle b, u-v\rangle\end{aligned} \tag{3.49}$$

for all $v \in \Xi$.

Remark 3.9. Reviewing Remark 2.2, we have

$$a(v,v) = \|v\|^2_{1,\Omega}, \quad \forall v \in H^1_0(\Omega).$$

This means that $a(u,v)$ is a coercive bilinear form on $H^1_0(\Omega)$. Recall from Lemma (2.4) that A is a positive operator. We can conclude that the solution

of (3.49) in Ξ is unique. Hence, (3.31)–(3.35) or (3.42)–(3.46) is equivalent to the original problem (3.3)–(3.8) or (3.9)–(3.14), and the solution has not been affected by the reduction.

Let $0 = x_0 < x_1 < \ldots < x_N = X$ be a regular partition of Ω, i.e., there is a positive constant $\varrho \in (0, 1)$ independent of h such that

$$\varrho h \leq x_j - x_{j-1}, \quad j = 1, 2, \ldots, N, \tag{3.50}$$

where N is a positive integer and $h = \max_{1 \leq j \leq N}(x_j - x_{j-1})$. For any positive integer M, let $k = T/M$, $t_m = mk$ for $m = -1, 0, 1, 2 \ldots, M$, and $J_m = (t_{m-1}, t_m]$ for $m = 0, 1, 2, \ldots, M$. For a function v on Q, denote $v(x, t_m)$ by v_m and define

$$\widetilde{v}(x,t) = \frac{t_m - t}{k} v(x, t_{m-1}) + \frac{t - t_{m-1}}{k} v(x, t_m), \ t \in J_m, \ m = 1, 2, \ldots, M.$$

Let V_h be the continuous piecewise linear element space with $v(X) = 0$ for $v \in V_h$, $K_h = \{v \in V_h : v \geq 0\}$, and

$$\Xi_{hk} = \left\{ v : v = \sum_{m=0}^{M} v_m \chi_m, \ v_m \in K_h \text{ for } m = 1, 2, \ldots, M \right\},$$

where $\chi_m(t)$ is the characteristic function of J_m.

Now we are in a position to define the finite element approximation to (3.49): Find $U \in \Xi_{hk}$ such that $U(x, 0) = 0$ and

$$\begin{aligned} \int_J \left(\left(\widetilde{U}_t, U - V \right) + a(U^\theta, U - V) \right) dt + \left\langle A\widetilde{U}, U - V \right\rangle \\ \leq \int_J (f, U - V) dt + \langle b, U - V \rangle \end{aligned} \tag{3.51}$$

for all $V \in \Xi_{hk}$, where $\theta \in [0, 1]$ and

$$U^\theta(x, t) = (1 - \theta)U(x, t) + \theta U(x, t - k).$$

Notice that the finite element solution U can be computed level by level in time. In fact, by writing $U = \sum_{m=0}^{M} U_m \chi_m$, taking $V = (1 - \chi_m)U + v\chi_m$

in (3.51) with $v \in K_h$, and using (2.13), we have the following equivalent problem: Find $U_1, U_2, \ldots, U_M$ in K_h such that for $m = 1, 2, \ldots, M$,

$$(3.52) \quad (\delta_k U_m, U_m - v) + a(U_{m-\theta}, U_m - v) + \sum_{j=1}^{m} w_{mj} \delta_k U_j(0)(U_m - v)(0) \\ \leq (f_m, U_m - v) + b_m(U_m - v)(0), \quad \forall v \in K_h,$$

where $U_0 = 0$ and

$$\delta_k U_m = \frac{U_m - U_{m-1}}{k}, \quad f_m(x) = \frac{1}{k}\int_{J_m} f(x,t)dt,$$
$$b_m = \frac{1}{k}\int_{J_m} b(t)dt, \quad w_{mj} = \frac{1}{k}\langle B\chi_j, \chi_m\rangle,$$
$$U_{m-\theta} = (1-\theta)U_m + \theta U_{m-1}.$$

For θ =0, 0.5, or 1.0, algorithm (3.52) is the implicit Euler scheme, the Crank-Nicholson scheme and the explicit Euler scheme, respectively.

Remark 3.10. Recalling Remark 3.4, we can define the approximations of $c(S,t)$ as follows

$$(3.53) \quad c_{hk}(S,t) = \begin{cases} S - K, & S \geq S^c_\infty,\ t > 0, \\ Ke^{-\alpha x - \beta\tau}\widetilde{U}(x,\tau) + S - K, & K \leq S \leq S^c_\infty,\ t > 0, \\ -\displaystyle\int_0^\tau E(x,\tau,s)\widetilde{U}(0,s)ds, & 0 < S < K,\ t > 0. \end{cases}$$

where $x = \ln(S/K)$, $\tau = \frac{\sigma^2}{2}(T_0 - t)$.

We need the following inverse estimate of V_h to show the existence, uniqueness and stability of the finite element solution U.

Lemma 3.1. *For any $v \in V_h$,*

$$(3.54) \quad \|v\|^2_{1,\Omega} \leq 8\varrho^{-2}h^{-2}\|v\|^2_{0,\Omega}.$$

Proof. For $v \in V_h$, we have

$$\|v\|^2_{0,\Omega} = \sum_{j=1}^{N} \frac{x_j - x_{j-1}}{6}(2v(x_j)^2 + v(x_j)v(x_{j-1}) + 2v(x_{j-1})^2)$$

$$\geq \sum_{j=1}^{N} \frac{x_j - x_{j-1}}{8}(v(x_j) - v(x_{j-1}))^2$$
$$\geq \frac{\varrho^2 h^2}{8} \sum_{j=1}^{N} \frac{(v(x_j) - v(x_{j-1}))^2}{x_j - x_{j-1}}$$
$$= \frac{\varrho^2 h^2}{8} \|v_x\|_{0,\Omega}^2 = \frac{\varrho^2 h^2}{8} \|v\|_{1,\Omega}^2.$$

where (3.50) was used. The proof is complete. □

Theorem 3.1. *The variational inequality (3.52) has a unique solution U. For $\theta \in [0, 1/2]$, U is absolutely stable; for $\theta \in (1/2, 1]$, if*

$$\frac{k}{h^2} \leq \gamma < \frac{\varrho^2}{4(2\theta - 1)}, \tag{3.55}$$

U is stable, where γ is a positive constant independent of h and k. In addition, we have

$$\left\|\widetilde{U}_t\right\|_{0,0}^2 + \max_{1 \leq m \leq M} \|U_m\|_{1,\Omega}^2 + \left|\widetilde{U}_t(0, \cdot)\right|_{-\frac{1}{4},J}^2 \leq C_{11} \left(\|U_0\|_{1,\Omega}^2 + \|f\|_{0,0}^2 + |b|_{\frac{1}{4},J}^2\right), \tag{3.56}$$

where C_{11} is a positive constant independent of k, h, U, but dependent on θ and λ.

Proof. Inequality (3.52) can be rewritten as a linear complementarity problem. It follows from (2.15) and (2.2) that the associated matrix is symmetric positive definite. Thus (3.52) has a unique solution. By Setting $v = U_{n-1}$ in (3.52) and summing on n from 1 through m for $m = 1, 2, \ldots, N$, we get

$$k \sum_{n=1}^{m} \|\delta_k U_n\|_{0,\Omega}^2 + k \sum_{n=1}^{m} a(U_{n-\theta}, \delta_k U_n) + \sum_{n=1}^{m} \sum_{j=1}^{n} \langle B\chi_j, \chi_n \rangle \delta_k U_n(0) \delta_k U_j(0)$$
$$\leq k \sum_{n=1}^{m} (f_n, \delta_k U_n) + k \sum_{n=1}^{m} b_n \delta_k U_n(0),$$

i.e.,

$$\int_0^{t_m} \left\|\widetilde{U}_t\right\|_{0,\Omega}^2 dt + \frac{k(1-2\theta)}{2} \int_0^{t_m} \left\|\widetilde{U}_t\right\|_{1,\Omega}^2 dt + \frac{1}{2}\|U_m\|_{1,\Omega}^2 - \frac{1}{2}\|U_0\|_{1,\Omega}^2 +$$
$$\int_0^{t_m} \widetilde{U}_t(0,t) B \widetilde{U}_t(0,t) dt \leq \int_0^{t_m} \left(f, \widetilde{U}_t\right) dt + \int_0^{t_m} b(t) \widetilde{U}_t(0,t) dt,$$

where we used the following identities

$$a(U_n, \delta_k U_n) = \frac{1}{2}\left(\|\delta_k U_n\|_{1,\Omega}^2 + \|U_n\|_{1,\Omega}^2 - \|U_{n-1}\|_{1,\Omega}^2\right),$$
$$a(U_{n-1}, \delta_k U_n) = \frac{1}{2}\left(-\|\delta_k U_n\|_{1,\Omega}^2 + \|U_n\|_{1,\Omega}^2 - \|U_{n-1}\|_{1,\Omega}^2\right).$$

By (2.15), (2.17) and Schwarz's inequality, we have

$$\int_0^{t_m} \widetilde{U}_t(0,t) B \widetilde{U}_t(0,t) dt \geq \frac{1}{\sqrt{2}} \left|\widetilde{U}_t\right|_{-\frac{1}{4},(0,t_m)}^2,$$
$$\int_0^{t_m} \left(f, \widetilde{U}_t\right) dt \leq \frac{1}{4\epsilon} \int_0^{t_m} \|f\|_{0,\Omega}^2 dt + \epsilon \int_0^{t_m} \left\|\widetilde{U}_t\right\|_{0,\Omega}^2 dt,$$
$$\int_0^{t_m} b(t) \widetilde{U}_t(0,t) dt \leq \frac{1}{\sqrt{2}} |b|_{\frac{1}{4},(0,t_m)}^2 + \frac{1}{2\sqrt{2}} \left|\widetilde{U}_t\right|_{-\frac{1}{4},(0,t_m)}^2,$$

where ϵ is any positive number. Hence,

$$(1-\epsilon)\int_0^{t_m} \left\|\widetilde{U}_t\right\|_{0,\Omega}^2 dt + \frac{k(1-2\theta)}{2} \int_0^{t_m} \left\|\widetilde{U}_t\right\|_{1,\Omega}^2 dt + \frac{1}{2}\|U_m\|_{1,\Omega}^2 +$$
$$\frac{1}{2\sqrt{2}} \left|\widetilde{U}_t\right|_{-\frac{1}{4},(0,t_m)}^2 \leq \frac{1}{2}\|U_0\|_{1,\Omega}^2 + \frac{1}{4\epsilon}\int_0^{t_m} \|f\|_{0,\Omega}^2 dt + \frac{1}{\sqrt{2}}|b|_{\frac{1}{4},J}^2,$$

where the following inequality was used:

$$|b|_{\frac{1}{4},(0,t_m)} \leq |b|_{\frac{1}{4},J}, \quad m = 1, 2, \ldots, N.$$

For $\theta \in [0, 1/2]$, estimate (3.56) follows from the above inequality with $\epsilon = 1/2$. For $\theta \in (1/2, 1]$, by Lemma 3.1,

$$\left(1 - \epsilon - 4(2\theta - 1)\varrho^{-2}\frac{k}{h^2}\right)\int_0^{t_m} \left\|\widetilde{U}_t\right\|_{0,\Omega}^2 dt + \frac{1}{2}\|U_m\|_{0,\Omega}^2$$
$$+ \frac{1}{2\sqrt{2}} \left|\widetilde{U}_t\right|_{-\frac{1}{4},(0,t_m)}^2 \leq \frac{1}{2}\|U_0\|_{0,\Omega}^2 + \frac{1}{4\epsilon}\int_0^{t_m} \|f\|_{0,\Omega}^2 dt + \frac{1}{\sqrt{2}}|b|_{\frac{1}{4},J}^2.$$

Hence, if (3.55) holds, estimate (3.56) follows from (2.2) and the above inequality with $\epsilon = 1 - 4(2\theta - 1)\varrho^{-2}\gamma$. □

3.4 Error Estimates

In this section we shall prove the main result of this chapter under the assumption that $u \in H^{2,1}((0,X) \times (0,T))$, which is reasonable in view of our earlier comments (see Remark 3.8).

Theorem 3.2. *Let u and U be solutions to (3.49) and (3.52). Then under the conditions of Theorem 3.1,*

$$\|u-U\|_{1,0}+\left|u(0,\cdot)-\widetilde{U}(0,\cdot)\right|_{\frac{1}{4},J}\le C\left(k^{\frac{1}{2}}+h\right), \tag{3.57}$$

where C is a positive constant independent of h, k and U but dependent on $\|u\|_{2,1}$, $\|f\|_{0,0}$, $\|b\|_{\frac{1}{4},J}$ (From now on we shall use C for such generic constants).

In order to prove this theorem, we need the following technical lemmas.

Lemma 3.2. *For any piecewise linear function (possibly discontinuous) v on J_n $(n=1,\ 2,\ \dots,N)$, the estimate*

$$|v|_{\frac{1}{4},J}\le Ck^{-\frac{1}{4}}|v|_{0,J} \tag{3.58}$$

holds.

Proof. Write a piecewise linear function v in the following form:

$$v(t)=a_n\frac{t_n-t}{k}+b_n\frac{t-t_{n-1}}{k}=\phi(x)+\psi(x),\quad t\in J_n,\quad n=1,2,\dots,N.$$

It is easy to see that

$$|\phi|_{0,J}+|\psi|_{0,J}\le 2|v|_{0,J}.$$

Hence if (3.58) holds for ϕ and ψ, then

$$|v|_{\frac{1}{4},J}\le|\phi|_{\frac{1}{4},J}+|\psi|_{\frac{1}{4},J}\le Ck^{-\frac{1}{4}}(|\phi|_{0,J}+|\psi|_{0,J})\le 2Ck^{-\frac{1}{4}}|v|_{0,J},$$

that is, (3.58) holds for v.

Now we prove that (3.58) is true for ϕ, and observe that the proof for ψ is the same. Direct calculation gives that

$$|\phi|^2_{0,J}=\frac{k}{3}\sum_{n=1}^{N}a_i^2. \tag{3.59}$$

It follows from Theorem 7.48 of [1] that

$$|\phi|^2_{\frac{1}{4},J}\le C\left(|\phi|^2_{0,J}+\int_0^T\int_0^T\frac{|\phi(t)-\phi(s)|^2}{|t-s|^{3/2}}dtds\right).$$

Clearly, we only need to estimate the integral term. We introduce the variable changes: $\xi = (t_n - t)/k$, $\eta = (t_m - s)/k$, and conclude

$$\begin{aligned}
&\int_0^T \int_0^T \frac{|\phi(t) - \phi(s)|^2}{|t-s|^{3/2}} dtds = \sum_{n=1}^{N} \sum_{m=1}^{N} \int_{t_{n-1}}^{t_n} \int_{t_{m-1}}^{t_m} \frac{|\phi(t) - \phi(s)|^2}{|t-s|^{3/2}} dtds \\
&= k^{\frac{1}{2}} \sum_{n=1}^{N} \sum_{m=1}^{N} \int_0^1 \int_0^1 \frac{|a_n \xi - a_m \eta|^2}{|n - m + \eta - \xi|^{3/2}} d\xi d\eta \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where I_1 the summation of diagonal terms ($n = m$), I_2 is the summation of sub-diagonal terms ($|n - m| = 1$) and I_3 is the summation of the other terms ($|n - m| \geq 2$), i.e.,

$$\begin{aligned}
I_1 &= k^{\frac{1}{2}} \sum_{n=1}^{N} \int_0^1 \int_0^1 a_n^2 |\eta - \xi|^{\frac{1}{2}} d\xi d\eta, \\
I_2 &= 2k^{\frac{1}{2}} \sum_{n=1}^{N-1} \int_0^1 \int_0^1 \frac{|a_n \xi - a_{n+1} \eta|^2}{|1 + \eta - \xi|^{3/2}} d\xi d\eta, \\
I_3 &= 2k^{\frac{1}{2}} \sum_{n=1}^{N-2} \sum_{m=n+2}^{N} \int_0^1 \int_0^1 \frac{|a_n \xi - a_m \eta|^2}{|n - m + \eta - \xi|^{3/2}} d\xi d\eta.
\end{aligned}$$

It follows from (3.59) that

$$I_1 = 3k^{-\frac{1}{2}} \int_0^1 \int_0^1 |\eta - \xi|^{\frac{1}{2}} d\xi d\eta \left(\frac{k}{3} \sum_{n=1}^{N} a_n^2 \right) = \frac{8}{5} k^{-\frac{1}{2}} |\phi|_{0,J}^2.$$

By Cauchy's inequality and (3.59),

$$\begin{aligned}
I_2 &\leq 2k^{\frac{1}{2}} \sum_{n=1}^{N-1} \int_0^1 \int_0^1 \frac{(\xi^2 + \eta^2)(a_n^2 + a_{n+1}^2)}{|1 + \eta - \xi|^{3/2}} d\xi d\eta \\
&\leq C k^{\frac{1}{2}} \int_0^1 \int_0^1 \frac{\xi^2 + \eta^2}{|1 + \eta - \xi|^{3/2}} d\xi d\eta \sum_{n=1}^{N} a_n^2 \\
&\leq C k^{-\frac{1}{2}} |\phi|_{0,J}^2
\end{aligned}$$

and

$$I_3 \leq 2k^{\frac{1}{2}} \sum_{n=1}^{N-2} \sum_{m=n+2}^{N} \int_0^1 \int_0^1 \frac{(\xi^2 + \eta^2)(a_n^2 + a_m^2)}{|m - n - 1|^{3/2}} d\xi d\eta$$

$$
\begin{aligned}
&= \frac{4}{3}k^{\frac{1}{2}} \sum_{n=1}^{N-2} \sum_{m=n+2}^{N} \frac{a_n^2}{|m-n-1|^{3/2}} + \frac{4}{3}k^{\frac{1}{2}} \sum_{n=1}^{N-2} \sum_{m=n+2}^{N} \frac{a_m^2}{|m-n-1|^{3/2}} \\
&= \frac{4}{3}k^{\frac{1}{2}} \sum_{n=1}^{N-2} \sum_{m=n+2}^{N} \frac{a_n^2}{|m-n-1|^{3/2}} + \frac{4}{3}k^{\frac{1}{2}} \sum_{m=3}^{N} \sum_{n=1}^{m-2} \frac{a_m^2}{|m-n-1|^{3/2}} \\
&\le \frac{8}{3}k^{\frac{1}{2}} \sum_{p=1}^{\infty} |p|^{-3/2} \sum_{n=1}^{N} a_n^2 \\
&\le Ck^{-\frac{1}{2}} |\phi|_{0,J}^2.
\end{aligned}
$$

To sum up, (3.58) holds for ϕ. □

Lemma 3.3.

$$
\left| U(0,\cdot) - \widetilde{U}(0,\cdot) \right|_{\frac{1}{4},J}^2 \le Ck \tag{3.60}
$$

Proof. It follows from Lemma 2.2 of [33] and (3.56) that

$$
k^{\frac{1}{2}} \left| \widetilde{U}_t(0,\cdot) \right|_{0,J}^2 \le C \left| \widetilde{U}_t(0,\cdot) \right|_{-\frac{1}{4},J}^2 \le C
$$

Thus,

$$
\sum_{n=1}^{N} |\delta_k U_n(0)|^2 = k^{-1} \left| \widetilde{U}_t(0,\cdot) \right|_{0,J}^2 \le Ck^{-\frac{3}{2}}.
$$

Since

$$
U(0,\cdot) - \widetilde{U}(0,\cdot) = \sum_{n=1}^{N} \delta_k U_n(0)\,(t_n - t)\,\chi_n,
$$

we have by Lemma 3.2

$$
\begin{aligned}
\left| U(0,\cdot) - \widetilde{U}(0,\cdot) \right|_{\frac{1}{4},J}^2 &\le Ck^{-\frac{1}{2}} \left| U(0,\cdot) - \widetilde{U}(0,\cdot) \right|_{0,J}^2 \\
&= \frac{C}{3}k^{\frac{5}{2}} \sum_{n=1}^{N} |\delta_k U_n(0)|^2 \le Ck.
\end{aligned}
$$

□

Let I_h be the interpolation operator from $C\left(\overline{\Omega}\right)$ to V_h. It is well known ([30]) that for $v \in H^2(\Omega) \cap H_E^1(\Omega)$,

$$
\|v - I_h v\|_{m,\Omega}^2 \le C_9 h^{4-2m} \|v\|_{2,\Omega}^2, \ m = 0, 1, \tag{3.61}
$$

where C_9 is a positive constant independent of h, v.

Lemma 3.4. *Let u be the solution to (3.49). Write*

$$\overline{u}(x,t) = \sum_{n=1}^{N} \overline{u}_n(x)\chi_n(t),$$

where $\overline{u}_n(x)$ denotes the Steklov average, i.e.,

$$\overline{u}_n(x) = \frac{1}{k}\int_{J_n} u(x,t)dt, \quad n = 1,2,\ldots,N.$$

Then $\overline{u} \in H^{2,0}(Q)$ and

$$\|u - I_h\overline{u}\|_{m,0}^2 \le C(k^{2-m} + h^{4-2m}), \quad m = 0,1. \tag{3.62}$$

Proof. It follows from direct calculation that $\overline{u} \in H^{2,0}(Q)$ and

$$\|\overline{u}\|_{2,0} \le \|u\|_{2,0}. \tag{3.63}$$

By using Schwarz's inequality twice, we get for any $t \in J_n$

$$\begin{aligned}
&|u(x,t) - \overline{u}_n(x)|^2 = \frac{1}{k^2}\left(\int_{J_n}(u(x,t) - u(x,\xi))d\xi\right)^2 \\
&\le \frac{1}{k^2}\int_{J_n} 1d\xi \int_{J_n}|u(x,t) - u(x,\xi)|^2 d\xi = \frac{1}{k}\int_{J_n}\left|\int_{\xi}^{t} u_t(x,\eta)d\eta\right|^2 d\xi \\
&\le \int_{J_n}\frac{t-\xi}{k}\int_{\xi}^{t}|u_t(x,\eta)|^2 d\eta d\xi \le \int_{J_n} k|u_t(x,\eta)|^2 d\eta.
\end{aligned}$$

Thus

$$\begin{aligned}
\|u - \overline{u}\|_{0,0}^2 &= \int_{\Omega}\sum_{n=1}^{N}\int_{J_n}(u(x,t) - \overline{u}_n(x))^2 dtdx \\
&\le \int_{\Omega}\sum_{n=1}^{N}\int_{J_n}\int_{J_n} k|u_t(x,\eta)|^2 d\eta dtdx = k^2\|u_t\|_{0,0}^2.
\end{aligned}$$

Hence we have

$$\|u - \overline{u}\|_{0,0}^2 \le Ck^2, \tag{3.64}$$

Similarly, we get by using (3.61)

$$\|\overline{u} - I_h\overline{u}\|_{m,0}^2 \le Ch^{4-2m}, \quad m = 0,1. \tag{3.65}$$

Thus (3.62) for $m = 0$ follows from the above estimates. Applying the interpolation inequality of Sobolev space (see Theorem 1.9.6 of [56]), we obtain

$$\|u(\cdot,t) - \overline{u}(\cdot,t)\|_{1,\Omega} \leq C\|u(\cdot,t) - \overline{u}(\cdot,t)\|_{2,\Omega}^{\frac{1}{2}}\|u(\cdot,t) - \overline{u}(\cdot,t)\|_{0,\Omega}^{\frac{1}{2}}, \quad \forall t \in [0,T].$$

Hence, by Schwarz's inequality , (3.63) and (3.64),

$$\|u - \overline{u}\|_{1,0}^2 \leq C\|u - \overline{u}\|_{2,0}\|u - \overline{u}\|_{0,0} \leq Ck.$$

By combining the above inequality and (3.65), we obtain (3.62) for $m = 1$. $\square$

Lemma 3.5.

$$|u(0,\cdot) - \overline{u}(0,\cdot)|_{\frac{1}{4},J}^2 \leq Ck. \tag{3.66}$$

Proof. Define

$$w(x,t) = \frac{t_n - t}{k}\overline{u}_{n-1}(x) + \frac{t - t_{n-1}}{k}\overline{u}_n(x), \ t \in J_n, \ n = 1, 2, \ldots, N.$$

We observe that $w \in H^{2,1}(Q)$. By using the same calculation as in the derivation of (3.64), we obtain for $m = 0, 2$

$$\|w\|_{m,\frac{m}{2}}^2 \leq C,$$
$$\|u - w\|_{m,\frac{m}{2}}^2 \leq Ck^{2-m}.$$

It follows from (2.9) that the above inequality holds for $m = 1$. Hence, by the trace estimate (2.10),

$$|u(0,\cdot) - w(0,\cdot)|_{\frac{1}{4},J}^2 \leq Ck. \tag{3.67}$$

Noticing that

$$\overline{u}(0,\cdot) - w(0,\cdot) = \sum_{n=1}^{N} \delta_k \overline{u}_n(0)\,(t_n - t)\,\chi_n,$$

we have by Lemma 3.2

$$|\overline{u}(0,\cdot) - w(0,\cdot)|_{\frac{1}{4},J}^2 \leq Ck^{-\frac{1}{2}}|\overline{u}(0,\cdot) - w(0,\cdot)|_{0,J}^2 \leq Ck^{\frac{5}{2}}\sum_{n=1}^{N}|\delta_k\overline{u}_n(0)|^2.$$

But by Theorem 7.48 of [1],

$$\begin{aligned}
\|w(0,\cdot)\|^2_{\frac{3}{4},J} &\geq C\left(|w(0,\cdot)|^2_{0,J} + \int_0^T\int_0^T \frac{|w(0,t)-w(0,s)|^2}{|t-s|^{5/2}}dtds\right)\\
&\geq C\sum_{n=1}^N \int_{J_n}\int_{J_n} \frac{|w(0,t)-w(0,s)|^2}{|t-s|^{5/2}}dtds\\
&= C\sum_{n=1}^N \int_{J_n}\int_{J_n} \left|\frac{w(0,t)-w(0,s)}{t-s}\right|^2 |t-s|^{-\frac{1}{2}}dtds\\
&= C\sum_{n=1}^N \int_{J_n}\int_{J_n} |\delta_k\overline{u}_n(0)|^2|t-s|^{-\frac{1}{2}}dtds\\
&\geq Ck^{\frac{3}{2}}\sum_{n=1}^N |\delta_k\overline{u}_n(0)|^2.
\end{aligned}$$

Thus,

$$|\overline{u}(0,\cdot)-w(0,\cdot)|^2_{\frac{1}{4},J} \leq Ck\|w(0,\cdot)\|^2_{\frac{3}{4},J}.$$

By the trace estimate (2.10) again,

$$|\overline{u}(0,\cdot)-w(0,\cdot)|^2_{\frac{1}{4},J} \leq Ck\|w\|^2_{2,1} \leq Ck.$$

Inequality (3.66) follows from the above estimate and (3.67). □

The proof of Theorem 3.2. For simplicity, we only give a proof for $\theta = 1$. It is not difficult to modify the proof for the case that $\theta \neq 1$.

Substituting $v = (1-\chi_n)\,u + \chi_n U_n$ in (3.49) and $V = (1-\chi_n)\,U + \chi_n I_h\overline{u}_n$ in (3.51), we have

$$\left(u_n - u_{n-1}, \frac{u_{n-1}+u_n}{2} - U_n\right) + \int_{J_n}(u(0,t)-U(0,t))Au(0,t)dt +$$
$$\int_{J_n} a(u,u-U)dt \leq \int_{J_n}(f,u-U)dt + \int_{J_n} b(t)(u(0,t)-U(0,t))dt,$$
$$(U_n - U_{n-1}, U_n - I_h\overline{u}_n) + \int_{J_n}(U(0,t)-I_h\overline{u}(0,t))A\widetilde{U}(0,t)dt +$$
$$\int_{J_n} a(U,U-I_h\overline{u})dt \leq \int_{J_n}(f,U-I_h\overline{u})dt + \int_{J_n} b(t)(U(0,t)-I_h\overline{u}(0,t))dt,$$

where the following identity was used:

$$\int_{J_n}(u_t, u-U_n)dt = \left(u_n - u_{n-1}, \frac{u_{n-1}+u_n}{2} - U_n\right).$$

Letting $e = u - U$ and adding the above inequalities, we obtain

$$(e_n - e_{n-1}, e_n) + \int_{J_n} \|e\|_{1,\Omega}^2 dt + \int_{J_n} \left(u - \widetilde{U}\right)(0,t) A \left(u - \widetilde{U}\right)(0,t) dt \leq$$
$$\frac{1}{2}\|u_n - u_{n-1}\|_0^2 + \int_{J_n} \left(\widetilde{U}_t, u - u_n\right) dt + \int_{J_n} \left(f + u_{xx} - \widetilde{U}_t, u - I_h \overline{u}\right) dt$$
$$+ \int_{J_n} a\left(e, u - I_h \overline{u}\right) dt + \int_{J_n} \left(u - \overline{u} + U - \widetilde{U}\right)(0,t) A \left(u - \widetilde{U}\right)(0,t) dt.$$

where $I_h \overline{u}(0,t) = \overline{u}_n(0)$ on J_n and the following identities were used:

$$\left(u_n - u_{n-1}, \frac{u_n + u_{n-1}}{2} - U_n\right) = (u_n - u_{n-1}, e_n) - \frac{1}{2}\|u_n - u_{n-1}\|_0^2,$$
$$(U_n - U_{n-1}, U_n - I_h \overline{u}_n) = -(U_n - U_{n-1}, e_n) + \int_{J_n} \left(\widetilde{U}_t, u_n - I_h \overline{u}\right) dt,$$
$$u(0,t) - U(0,t) = \left(u - \widetilde{U}\right)(0,t) + \left(\widetilde{U} - U\right)(0,t),$$
$$U(0,t) - I_h \overline{u}(0,t) = -\left(u - \widetilde{U}\right)(0,t) + \left(U - \widetilde{U} + u - I_h u\right)(0,t),$$
$$a(U, U - I_h \overline{u}) = -a(U, e) - a(e, u - I_h \overline{u}) + a(u, u - I_h \overline{u}),$$
$$a(u, u - I_h \overline{u}) = -(u(0,t) - I_h \overline{u}(0,t))(Au(0,t) - b(t)) - (u_{xx}, u - I_h \overline{u}).$$

Since

$$e_0 = 0, \quad (e_n - e_{n-1}, e_n) = \frac{1}{2}\left(\|e_n - e_{n-1}\|_{0,\Omega}^2 + \|e_n\|_{0,\Omega}^2 - \|e_{n-1}\|_{0,\Omega}^2\right),$$

we have by summation on n from 1 through N

$$\int_J \|e\|_{1,\Omega}^2 dt + \left\langle A\left(u - \widetilde{U}\right), u - \widetilde{U}\right\rangle \leq R_1 + R_2 + R_3 + R_4 + R_5,$$

where

$$R_1 = \frac{1}{2}\sum_{n=1}^{N} \|u_n - u_{n-1}\|_{0,\Omega}^2$$
$$R_2 = \sum_{n=1}^{N} \int_{J_n} \left(\widetilde{U}_t, u - u_n\right) dt,$$
$$R_3 = \int_J \left(f + u_{xx} - \widetilde{U}_t, u - I_h \overline{u}\right) dt,$$
$$R_4 = \int_J a(e, u - I_h \overline{u}) dt,$$
$$R_5 = \left\langle A\left(u - \widetilde{U}\right), u - \overline{u} + U - \widetilde{U}\right\rangle.$$

Thus, by (2.2),

$$\|e\|_{1,0}^2 + C_3 \left\|u - \widetilde{U}\right\|_{\frac{1}{4},J}^2 \le R_1 + R_2 + R_3 + R_4 + R_5. \tag{3.68}$$

For R_1 and R_2, we have by Schwarz's inequality and (3.56)

$$R_1 = \frac{1}{2}\sum_{n=1}^{N}\int_\Omega \left(\int_{J_n} u_t(x,t)dt\right)^2 dx \le \frac{k}{2}\|u\|_{0,1}^2, \tag{3.69}$$

$$\begin{aligned} R_2 &= \sum_{n=1}^{N}\int_{J_n}\left(\widetilde{U}_t, u - u_n\right) dt = \sum_{n=1}^{N}\int_\Omega\int_{J_n}\int_{t_n}^{t}\widetilde{U}_t(x,t)u_t(x,s)dsdtdx \\ &\le \int_\Omega\left(\sum_{n=1}^{N}\int_{J_n}\left|\widetilde{U}_t(x,t)\right|dt\int_{J_n}|u_t(x,s)|ds\right)dx \\ &\le k\left(\left\|\widetilde{U}_t\right\|_{0,0}^2 + \|u_t\|_{0,0}^2\right) \le Ck. \end{aligned} \tag{3.70}$$

By using Schwarz's inequality, (3.56) and (3.62), we get

$$|R_3| \le \|f + u_{xx} - \widetilde{U}_t\|_{0,0}\|u - I_h\overline{u}\|_{0,0} \le C(k + h^2), \tag{3.71}$$

$$|R_4| \le \|e\|_{1,0}\|u - I_h\overline{u}\|_{1,0} \le \frac{C_6}{2}\|e\|_{1,0}^2 + C(k + h^2). \tag{3.72}$$

It follows from Lemma 2.4 that

$$|R_5| \le \frac{C_3}{2}\left|u(0,\cdot) - \widetilde{U}(0,\cdot)\right|_{\frac{1}{4},J}^2 + C\left|u(0,\cdot) - \overline{u}(0,\cdot) + U(0,\cdot) - \widetilde{U}(0,\cdot)\right|_{\frac{1}{4},J}^2.$$

Hence, by (3.60) and (3.66),

$$|R_5| \le \frac{C_3}{2}\left|u(0,\cdot) - \widetilde{U}(0,\cdot)\right|_{\frac{1}{4},J}^2 + Ck. \tag{3.73}$$

Combining (3.68)–(3.73), we complete the proof of (3.57). □

Next we estimate the error between the call option price $c(S,t)$ and its approximation $c_{hk}(S,t)$ defined in (3.53).

Theorem 3.3. *Let $e_{hk} = c - c_{hk}$. Then*

$$\max_{1\le n\le N}\|e_{hk}(\cdot,t_n)\|_{0,(K,S_\infty)} + \|e_{hk}\|_{H^{1,0}((K,S_\infty)\times(0,T_0))} \le C\left(k^{\frac{1}{2}} + h\right), \tag{3.74}$$

$$\left|\frac{\partial^{i+j}e_{hk}(S,t)}{\partial S^i\partial t^j}\right| \le CM(x,t)\left(k^{\frac{1}{2}} + h\right), \quad \forall 0 < S < K,\ 0 < t < T_0, \tag{3.75}$$

$$\|e_{hk}\|_{H^{1,\frac{1}{2}}((0,K)\times(0,T_0))} \le C\left(k^{\frac{1}{2}} + h\right), \tag{3.76}$$

where $x = \ln(S/K)$, $\tau = \frac{\sigma^2}{2}(T_0 - t)$, i *and* j *are any nonnegative integer and*

$$M(x,t) = |x|^{-\frac{1}{2}-i-2j} e^{-\frac{x^2}{8\tau}-\alpha x}.$$

Proof. Estimate (3.74) follows from (3.57). We have from Remark 3.4 and Remark 3.10 that for $0 < S < K$, $0 < t < T_0$,

$$e_{hk}(S,t) = \int_0^\tau E(x,\tau,s)\left(u(0,s) - \widetilde{U}(0,s)\right) ds.$$

Since $E(x,\tau,s)$ is smooth in $(-\infty, 0) \times (0,T)$ and

$$\left|\frac{\partial^{i+j} E(x,\tau,s)}{\partial S^i \partial t^j}\right| \le C \begin{cases} |x|(\tau - s)^{-(3+i)/2-j} e^{-\frac{x^2}{6(\tau-s)}}, & \text{if } i \text{ is even}, \\ (\tau - s)^{-(2+i)/2-j} e^{-\frac{x^2}{6(\tau-s)}}, & \text{if } i \text{ is odd}, \end{cases}$$

we obtain by Hölder's inequality

$$\begin{aligned} \left|\frac{\partial^{i+j} e_{hk}(S,t)}{\partial S^i \partial t^j}\right| &\le C \left(\int_0^\tau \left|\frac{\partial^{i+j} E(x,\tau,s)}{\partial x^i \partial t^j}\right|^{\frac{4}{3}} ds\right)^{\frac{3}{4}} \left\|u(0,\cdot) - \widetilde{U}(0,\cdot)\right\|_{L^4(J)} \\ &\le CM(x,t) \left\|u(0,\cdot) - \widetilde{U}(0,\cdot)\right\|_{L^4(J)} \end{aligned}$$

Recalling the embedding theorem of Sobolev spaces (Theorem 7.58 of [1]), we have

$$\left\|u(0,\cdot) - \widetilde{U}(0,\cdot)\right\|_{L^4(J)} \le C \left|u(0,\cdot) - \widetilde{U}(0,\cdot)\right|_{\frac{1}{4},J}.$$

Hence (3.75) holds. It follows from the results of §4.15.1 of [56] that

$$\|e_{hk}\|_{H^{1,\frac{1}{2}}((K/2,K)\times(0,T))} \le C\left(|e_{hk}(K/2,\cdot)|_{\frac{1}{4},J} + |e_{hk}(K,\cdot)|_{\frac{1}{4},J}\right).$$

By (3.57) and (3.75), we obtain

$$\begin{aligned} \|e_{hk}\|_{H^{1,\frac{1}{2}}((K/2,K)\times(0,T_0))} &\le C\left(k^{\frac{1}{2}} + h\right), \\ \|e_{hk}\|_{H^{1,\frac{1}{2}}((0,K/2)\times(0,T_0))} &\le C\left(k^{\frac{1}{2}} + h\right). \end{aligned}$$

These two estimates lead to (3.76). □

Now we consider a put option. Recalling Remark 3.5, we can define the approximations of $p(S,t)$ as follows:

$$(3.77) \quad p_{hk}(S,t) = \begin{cases} K - S, & 0 \le S < S_\infty,\ t > 0, \\ Ke^{-\alpha x - \beta\tau} U(-x,\tau) + K - S, & S_\infty \le S \le K,\ t > 0, \\ \int_0^\tau E(-x,\tau,s)\widetilde{U}(0,s) ds, & S > K,\ t > 0, \end{cases}$$

where $x = \ln(S/K)$, $\tau = \frac{\sigma^2}{2}(T_0 - t)$.

Theorem 3.4. *Let $e_{hk} = p - p_{hk}$. Then*

$$\max_{1\le n\le N} \|e_{hk}(\cdot, t_n)\|_{0,(S_\infty,K)} + \|e_{hk}\|_{H^{1,0}((S_\infty,K)\times(0,T))} \le C\left(k^{\frac{1}{2}} + h\right), \tag{3.78}$$

$$\left|\frac{\partial^{i+j} e_{hk}(S,t)}{\partial S^i \partial t^j}\right| \le C M_{ij}(x,t)\left(k^{\frac{1}{2}} + h\right), \quad \forall S > K,\ t > 0, \tag{3.79}$$

$$\|e_{hk}\|_{H^{1,\frac{1}{2}}((K,K')\times(0,T))} \le C\left(k^{\frac{1}{2}} + h\right), \tag{3.80}$$

where $x = \ln(S/K)$, $\tau = \frac{\sigma^2}{2}(T_0 - t)$, i and j are any nonnegative integer, $K' > K$, and $M_{ij}(x,t)$ is defined as in Theorem 3.3.

Remark 3.11. (3.75) and (3.79) imply that approximations to a call and to a put are uniform in any compact subset of $[0, K) \times [0, T]$ and $(K, \infty) \times [0, T]$, respectively.

3.5 Numerical Examples

In this section, we shall illustrate our considerations with some specific examples and compare with other approaches. In the following, we shall use the transform $\tau = T_0 - t$ instead of $T_0 - t = \frac{2\tau}{\sigma^2}$ in (3.17). Our software program was written in C++ and run on a personal computer with a Pentium III 500MHZ processor.

Example 3.1. In this example, we examine the convergence rate of our algorithm. Recalling Remark 3.6, we only consider American call options.

Numerical experiments show that the best approximations are obtained when the step sizes in x and in time t are almost equal. In these case, we may suppose that

$$u(x,t) \approx u_{hk}(x,t) + Ch^\nu$$

in $L^2(\Omega)$, $L^\infty(\Omega)$, or $H^1(\Omega)$ for any fixed time t, where $\Omega = (0, X_c)$, $u_{hk}(x,t)$ is $U(x,t)$ for mesh sizes h and k and C is a constant independent of h and k. Hence we have

$$\nu \approx \nu_h = \frac{\ln \frac{\|E_h(\cdot,t)\|}{\left\|E_{h/2}(\cdot,t)\right\|}}{\ln 2},$$

where $\|\cdot\|$ is the norm of $L^2(\Omega)$, $L^\infty(\Omega)$, or $H^1(\Omega)$ and

$$E_h(x,t) = u_{\frac{h}{2}\frac{k}{2}}(x,t) - u_{hk}(x,t).$$

We have the same argument about the convergence of $\widetilde{U}(0,t)$ in $L^2(\Omega)$ or $L^\infty(\Omega)$ and use the following notation

$$e_h(t) = \widetilde{u}_{\frac{h}{2}\frac{k}{2}}0,t) - \widetilde{u}_{hk}(0,t).$$

Consider the American call option on a stock with $T = 12$ months, $K =$ \$100 and two groups of other parameters (see Table 3.1) which are two special cases.

σ	r	d	S_∞^c	X_c
0.2	0.08	0.12	133.333334	0.28768
0.4	0.15	0.05	500.0	1.60944

Table 3.1: Parameters

We see from Table 3.1 that the variational inequality (3.31)–(3.35) is solved in $[0, 0.28768] \times [0,1]$ and $[0, 1.60944] \times [0,1]$, which are very narrow as mentioned earlier.

In Table 3.2–3.4 and Table 3.6–3.8, we display $\|E_h(\cdot,t)\|$ and ν_h, where Euler denotes the Euler scheme and CN denotes the Crank-Nicholson scheme. In Table 3.5 and Table 3.9, we do the same thing for $\widetilde{U}(0,t)$. The results show that the convergence rates of U are of order 1 in $L^2(\Omega)$, $L^\infty(\Omega)$ and $H^1(\Omega)$ for Euler scheme. But Crank-Nicholson scheme is of order 1 for the first group of parameters and of order more than 1 for the seond group of parameters. This implies that the local regularity of option prices depends on parameters. The convergence rates of $\widetilde{U}(0,t)$ is of order 1 and order 0.5 in $L^2(0,T_0)$, $L^\infty(0,T_0)$, respectively. They are better than the theoretical rate $O\left(k^{\frac{1}{2}} + h\right)$.

h	$\|E(\cdot, 0.25)\|_{0,\Omega}$	ν_h	$\|E(\cdot, 0.25)\|_{1,\Omega}$	ν_h
6.2540e−3	1.2414e−5	0.7532	2.4923e−3	1.0004
3.1270e−3	7.3650e−6	0.8482	1.2459e−3	1.0011
1.5635e−3	4.0911e−6	0.9021	6.2247e−4	0.9998
7.8174e−4	2.1892e−6	0.9332	3.1127e−4	0.9996
3.9087e−4	1.1465e−6	0.9510	1.5567e−4	0.9995
1.9544e−4	5.9306e−7	0.9628	7.7864e−5	0.9998

Table 3.2: Euler: $\sigma = 0.2$, $r = 0.08$, $d = 0.12$, 3 months to the expiration

h	$\|E(\cdot, 1.0)\|_{0,\Omega}$	ν_h	$\|E(\cdot, 1.0)\|_{1,\Omega}$	ν_h
6.2540e−3	9.5652e−6	0.7924	2.4236e−3	0.9953
3.1270e−3	5.5227e−6	0.8795	1.2158e−3	1.0014
1.5635e−3	3.0019e−6	0.9250	6.0730e−4	0.9988
7.8174e−4	1.5811e−6	0.9513	3.0391e−4	1.0003
3.9087e−4	8.1771e−7	0.9655	1.5192e−4	0.9997
1.9544e−4	4.1873e−7	0.9745	7.5979e−5	1.0001

Table 3.3: Euler: $\sigma = 0.2$, $r = 0.08$, $d = 0.12$, 12 months to the expiration

h	$\|E(\cdot, 0.25)\|_{L^\infty(\Omega)}$	ν_h	$\|E(\cdot, 1.0)\|_{L^\infty(\Omega)}$	ν_h
6.2540e−3	6.6243e−5	0.9389	4.5044e−5	0.9832
3.1270e−3	3.4554e−5	0.9371	2.2785e−5	0.9728
1.5635e−3	1.8048e−5	0.9446	1.1609e−5	0.9719
7.8174e−4	9.3771e−6	0.9538	5.9189e−6	0.9745
3.9087e−4	4.8412e−6	0.9613	3.0123e−6	0.9774
1.9544e−4	2.4864e−6	0.9679	1.5299e−6	0.9807

Table 3.4: Euler: convergence in L^∞-norm, $\sigma = 0.2$, $r = 0.08$, $d = 0.12$,

k	$\|e_h\|_{L^2(0,T_0)}$	ν_h	$\|e_h\|_{L^\infty(0,T_0)}$	ν_h
$6.2500e{-3}$	$8.4220e{-5}$	0.9290	$1.3279e{-3}$	0.5035
$3.1250e{-3}$	$4.4234e{-5}$	0.9527	$9.3671e{-4}$	0.5018
$1.5625e{-3}$	$2.2853e{-5}$	0.9528	$6.6155e{-4}$	0.5009
$7.8125e{-4}$	$1.1806e{-5}$	0.9559	$4.6749e{-4}$	0.5004
$3.9063e{-4}$	$6.0866e{-6}$	0.9593	$3.3047e{-4}$	0.5002
$1.9531e{-4}$	$3.1303e{-6}$	0.9632	$2.3364e{-4}$	0.5001

Table 3.5: Euler: Convergence of $\widetilde{U}(0,t)$, $\sigma = 0.2$, $r = 0.08$, $d = 0.12$,

h	$\|E(\cdot, 0.25)\|_{0,\Omega}$	ν_h	$\|E(\cdot, 0.25)\|_{1,\Omega}$	ν_h
$6.2540e{-3}$	$3.3594e{-6}$	2.1632	$2.4783e{-3}$	1.0005
$3.1270e{-3}$	$7.5002e{-7}$	2.2649	$1.2387e{-3}$	0.9996
$1.5635e{-3}$	$1.5606e{-7}$	2.0603	$6.1954e{-4}$	1.0002
$7.8174e{-4}$	$3.7418e{-8}$	1.0919	$3.0972e{-4}$	0.9990
$3.9087e{-4}$	$1.7554e{-8}$	0.9218	$1.5497e{-4}$	0.9995
$1.9544e{-4}$	$9.2663e{-9}$	1.0381	$7.7516e{-5}$	0.9998

Table 3.6: CN: $\sigma = 0.2$, $r = 0.08$, $d = 0.12$, 3 months to the expiration

h	$\|E(\cdot, 0.25)\|_{0,\Omega}$	ν_h	$\|E(\cdot, 0.25)\|_{1,\Omega}$	ν_h
$6.2540e{-3}$	$3.4109e{-6}$	2.1349	$2.4221e{-3}$	0.9954
$3.1270e{-3}$	$7.7659e{-7}$	2.0733	$1.2149e{-3}$	1.0015
$1.5635e{-3}$	$1.8453e{-7}$	2.2545	$6.0683e{-4}$	0.9989
$7.8174e{-4}$	$3.8673e{-8}$	2.0677	$3.0364e{-4}$	1.0004
$3.9087e{-4}$	$9.2249e{-9}$	1.3723	$1.5178e{-4}$	0.9998
$1.9544e{-4}$	$3.5633e{-9}$	0.9805	$7.5901e{-5}$	1.0000

Table 3.7: CN: $\sigma = 0.2$, $r = 0.08$, $d = 0.12$, 12 months to the expiration

h	$\|E(\cdot, 0.25)\|_{L^\infty(\Omega)}$	ν_h	$\|E(\cdot, 1.0)\|_{L^\infty(\Omega)}$	ν_h
6.2540e−3	2.3047e−5	2.1380	1.6819e−5	2.0656
3.1270e−3	5.2361e−6	2.2772	4.0176e−6	2.1291
1.5635e−3	1.0802e−6	1.9457	9.1844e−7	2.2891
7.8174e−4	2.8042e−7	1.1827	1.8791e−7	1.9919
3.9087e−4	1.2354e−7	1.2162	4.7242e−8	1.2414
1.9544e−4	5.3173e−8	1.2227	1.9981e−8	1.2297

Table 3.8: CN: convergence in L^∞-norm, $\sigma = 0.2$, $r = 0.08$, $d = 0.12$,

k	$\|e_h\|_{L^2(0,T_0)}$	ν_h	$\|e_h\|_{L^\infty(0,T_0)}$	ν_h
6.2500e−3	9.5936e−5	0.9596	1.6449e−3	0.4481
3.1250e−3	4.9332e−5	0.9768	1.2057e−3	0.4745
1.5625e−3	2.5065e−5	0.9873	8.6779e−4	0.4875
7.8125e−4	1.2643e−5	0.9933	6.1896e−4	0.4939
3.9063e−4	6.3512e−6	0.9966	4.3953e−4	0.4970
1.9531e−4	3.1831e−6	0.9983	3.1143e−4	0.4986

Table 3.9: CN: Convergence of $\widetilde{U}(0,t)$, $\sigma = 0.2$, $r = 0.08$, $d = 0.12$,

h	$\|E(\cdot, 0.25)\|_{0,\Omega}$	ν_h	$\|E(\cdot, 0.25)\|_{1,\Omega}$	ν_h
6.2381e−3	2.9392e−5	0.9694	1.0770e−3	0.9994
3.1191e−3	1.5011e−5	0.9846	5.3875e−4	0.9999
1.5595e−3	7.5862e−6	0.9923	2.6939e−4	0.9999
7.7977e−4	3.8135e−6	0.9961	1.3471e−4	1.0000
3.8988e−4	1.9119e−6	0.9981	6.7356e−5	1.0000
1.9494e−4	9.5722e−7	0.9990	3.3679e−5	1.0000

Table 3.10: Euler: $\sigma = 0.4$, $r = 0.15$, $d = 0.05$, 3 months to the expiration

h	$\|\|E(\cdot, 0.25)\|\|_{0,\Omega}$	ν_h	$\|\|E(\cdot, 0.25)\|\|_{1,\Omega}$	ν_h
$6.2381e{-3}$	$2.1455e{-5}$	0.9772	$6.1540e{-4}$	1.0000
$3.1191e{-3}$	$1.0899e{-5}$	0.9886	$3.0770e{-4}$	0.9997
$1.5595e{-3}$	$5.4927e{-6}$	0.9943	$1.5389e{-4}$	0.9992
$7.7977e{-4}$	$2.7572e{-6}$	0.9971	$7.6987e{-5}$	1.0003
$3.8988e{-4}$	$1.3813e{-6}$	0.9986	$3.8485e{-5}$	0.9998
$1.9494e{-4}$	$6.9133e{-7}$	0.9995	$1.9245e{-5}$	1.0000

Table 3.11: Euler: $\sigma = 0.4$, $r = 0.15$, $d = 0.05$, 12 months to the expiration

h	$\|\|E(\cdot, 0.25)\|\|_{L^\infty(\Omega)}$	ν_h	$\|\|E(\cdot, 1.0)\|\|_{L^\infty(\Omega)}$	ν_h
$6.2381e{-3}$	$5.7066e{-5}$	0.9905	$3.5295e{-5}$	1.0016
$3.1191e{-3}$	$2.8721e{-5}$	0.9951	$1.7628e{-5}$	1.0008
$1.5595e{-3}$	$1.4410e{-5}$	0.9975	$8.8091e{-6}$	1.0004
$7.7977e{-4}$	$7.2174e{-6}$	0.9987	$4.4034e{-6}$	1.0002
$3.8988e{-4}$	$3.6119e{-6}$	0.9994	$2.2014e{-6}$	1.0002
$1.9494e{-4}$	$1.8068e{-6}$	0.9997	$1.1006e{-6}$	1.0005

Table 3.12: Euler: convergence in L^∞-norm, $\sigma = 0.4$, $r = 0.15$, $d = 0.05$

k	$\|\|e_h\|\|_{L^2(0,T_0)}$	ν_h	$\|\|e_h\|\|_{L^\infty(0,T_0)}$	ν_h
$6.2500e{-3}$	$1.3102e{-4}$	0.9777	$2.6458e{-3}$	0.5008
$3.1250e{-3}$	$6.6532e{-5}$	0.9766	$1.8698e{-3}$	0.5004
$1.5625e{-3}$	$3.3811e{-5}$	0.9761	$1.3218e{-3}$	0.5002
$7.8125e{-4}$	$1.7187e{-5}$	0.9762	$9.3454e{-4}$	0.5001
$3.9063e{-4}$	$8.7369e{-6}$	0.9765	$6.6077e{-4}$	0.5001
$1.9531e{-4}$	$4.4403e{-6}$	0.9769	$4.6722e{-4}$	0.5000

Table 3.13: Euler: convergence of $\widetilde{U}(0,t)$, $\sigma = 0.4$, $r = 0.15$, $d = 0.05$

h	$\|E(\cdot, 0.25)\|_{0,\Omega}$	ν_h	$\|E(\cdot, 0.25)\|_{1,\Omega}$	ν_h
$6.2381e{-3}$	$3.1176e{-6}$	1.9948	$1.6071e{-3}$	0.9805
$3.1191e{-3}$	$7.8222e{-7}$	1.9957	$8.1445e{-4}$	0.9866
$1.5595e{-3}$	$1.9614e{-7}$	1.9966	$4.1103e{-4}$	0.9904
$7.7977e{-4}$	$4.9150e{-8}$	1.9977	$2.0689e{-4}$	0.9933
$3.8988e{-4}$	$1.2307e{-8}$	1.9980	$1.0393e{-4}$	0.9952
$1.9494e{-4}$	$3.0810e{-9}$	1.9936	$5.2138e{-5}$	0.9966

Table 3.14: CN: $\sigma = 0.4$, $r = 0.15$, $d = 0.05$, 3 months to the expiration

h	$\|E(\cdot, 0.25)\|_{0,\Omega}$	ν_h	$\|E(\cdot, 0.25)\|_{1,\Omega}$	ν_h
$6.2381e{-3}$	9.2770e$-$7	2.0078	$6.0838e{-4}$	0.9994
$3.1191e{-3}$	$2.3067e{-7}$	2.0162	$3.0433e{-4}$	0.9986
$1.5595e{-3}$	$5.7026e{-8}$	2.0244	$1.5231e{-4}$	0.9987
$7.7977e{-4}$	$1.4017e{-8}$	1.9810	$7.6227e{-5}$	1.0001
$3.8988e{-4}$	$3.5507e{-9}$	2.0103	$3.8112e{-5}$	0.9990
$1.9494e{-4}$	8.8135e-10	1.4021	$1.9069e{-5}$	1.0000

Table 3.15: CN: $\sigma = 0.4$, $r = 0.15$, $d = 0.05$, 12 months to the expiration

h	$\|E(\cdot, 0.25)\|_{L^\infty(\Omega)}$	ν_h	$\|E(\cdot, 1.0)\|_{L^\infty(\Omega)}$	ν_h
$6.2381e{-3}$	$4.4953e{-5}$	1.5172	$3.0975e{-6}$	1.9967
$3.1191e{-3}$	$1.5705e{-5}$	1.5143	$7.7613e{-7}$	2.2196
$1.5595e{-3}$	$5.4978e{-6}$	1.5112	$1.6664e{-7}$	1.9962
$7.7977e{-4}$	$1.9288e{-6}$	1.5084	$4.1768e{-8}$	1.9330
$3.8988e{-4}$	$6.7796e{-7}$	1.5062	$1.0938e{-8}$	2.1506
$1.9494e{-4}$	$2.3866e{-7}$	1.5043	$2.4635e{-9}$	1.1797

Table 3.16: CN: convergence in L^∞-norm, $\sigma = 0.4$, $r = 0.15$, $d = 0.05$

k	$\|e_h\|_{L^2(0,T_0)}$	ν_h	$\|e_h\|_{L^\infty(0,T_0)}$	ν_h
$6.2500e{-3}$	$1.9973e{-4}$	0.9858	$3.4550e{-3}$	0.4858
$3.1250e{-3}$	$1.0085e{-4}$	0.9922	$2.4673e{-3}$	0.4925
$1.5625e{-3}$	$5.0701e{-5}$	0.9957	$1.7537e{-3}$	0.4960
$7.8125e{-4}$	$2.5426e{-5}$	0.9977	$1.2435e{-3}$	0.4978
$3.9063e{-4}$	$1.2733e{-5}$	0.9987	$8.8058e{-4}$	0.4988
$1.9531e{-4}$	$6.3722e{-6}$	0.9993	$6.2318e{-4}$	0.4993

Table 3.17: CN: convergence of $\widetilde{U}(0,t)$, $\sigma = 0.4$, $r = 0.15$, $d = 0.05$

The early exercise prices (the free boundaries) are shown in Figure 3.1 and Figure 3.2, which also shows that our algorithm converges quite rapidly. They are monotone as expected.

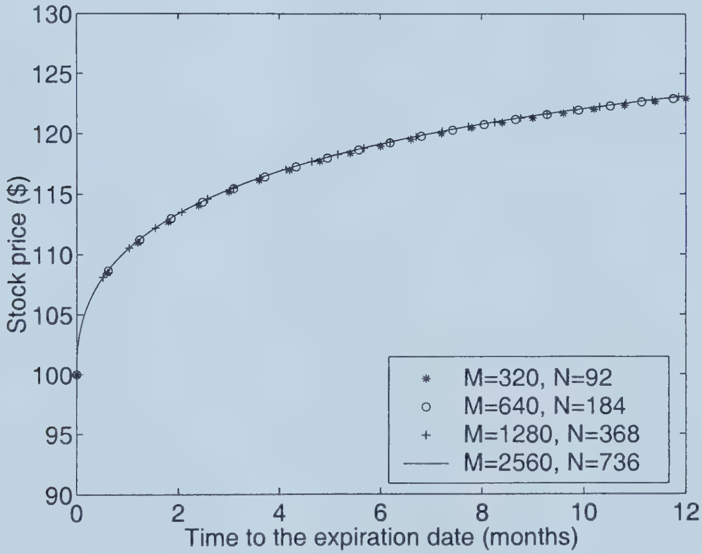

Figure 3.1: Early exercise prices: $\sigma = 0.2$, $r = 0.08$, $d = 0.12$

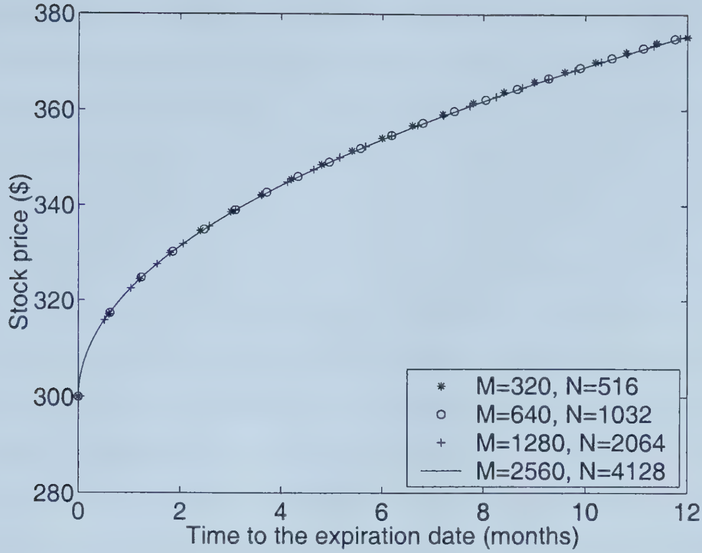

Figure 3.2: Early exercise prices: $\sigma = 0.4$, $r = 0.15$, $d = 0.05$

Example 3.2. As in [40] and [25], we consider American put options with K = \$35, \$40, or \$45, σ = 0.2, 0.3, or 0.4, and T = 1, 4, or 7 months. For fixed spot stock price S = \$40, interest rate $r = 0.0488$ and dividend $d = 0.0$, there are 27 combinations. We take the option prices computed by the binomial method with 10000 time steps as "exact".

We first examine the approximate option prices which are displayed in Table 3.18. BIN1 and BIN2 are the binomial methods with 10000 time steps and 150 time steps, respectively. FEM1 and FEM2 are the Euler scheme of our method with $M = 70$ plus the Richardson's extrapolation and the Crank-Nicholson scheme of our method with $M = 140$, respectively. E23I is the method of Eq.23/integration in [25] (which possesses the best accuracy in Table II of [25] and its CPU time has been adjusted according to the binomial method with 150 time steps). CPU times in seconds and the root of the mean squared errors (RMSE) are also presented in Table 3.18.

In order to compare different methods, as suggested in [25], we use the RMSE ratio and the CPU ratio. We recall that the RMSE ratio is the ratio between the RMSE of a method and the RMSE of the binomial method with

150 time steps and the CPU ratio is defined in the same way. The CPU ratio should not vary much between equivalent computers. We display these two ratios in Table 3.20. Data for other methods in Table 3.20 are taken from Table III of [25]. The number of steps is 200 for the finite difference method in both t and S and 150 for the accelerated binomial method. We refer the interested reader to [25] for the methods such as Eq(23)/integration and Eq(23)/four-point and to [40] for the recursive method with an extrapolation.

Next we examine the approximate hedge ratios which are displayed in Table 3.19. BIN1 and BIN2 are the binomial methods with 10000 time steps and 150 time steps, respectively. FEM1 and FEM2 are the Euler scheme of our method with $M = 140$ plus the Richardson's extrapolation (Euler) and the Crank-Nicholson scheme of our method with $M = 140$ plus the Richardson's extrapolation(CN), respectively. E23F is the method, Eq.23/four-point in [25] (E23F) (its CPU time has also been adjusted according to the binomial method with 150 time steps). CPU times in seconds and the root of mean squared errors (RMSE) are also presented in Table 3.19. Similarly, RMSE ratios and CPU ratios for hedge ratios are presented in Table 3.21.

All results in Table 3.18–Table 3.21 show that our methods possess the best accuracy and are also very fast.

σ	K	T	BIN1	BIN2	FEM1	FEM2	E23I
0.2	35	1	0.00620	0.00606	0.00621	0.00618	0.0062
		4	0.20039	0.19950	0.20031	0.20033	0.2005
		7	0.43285	0.43404	0.43274	0.43277	0.4332
	40	1	0.85231	0.85133	0.85165	0.85179	0.8528
		4	1.57986	1.57835	1.57961	1.57977	1.5813
		7	1.99048	1.98868	1.99034	1.99042	1.9915
	45	1	5.00000	5.00000	5.00000	5.00000	5.0020
		4	5.08834	5.08864	5.08764	5.08836	5.0911
		7	5.26703	5.26771	5.26632	5.26703	5.2647
0.3	35	1	0.07745	0.07760	0.07742	0.07736	0.0774
		4	0.69761	0.69935	0.69735	0.69746	0.6978
		7	1.21988	1.22394	1.21967	1.21978	1.2203
	40	1	1.31015	1.30851	1.30918	1.30897	1.3104
		4	2.48264	2.47997	2.48223	2.48250	2.4837
		7	3.16968	3.16656	3.16943	3.16961	3.1705
	45	1	5.05974	5.06005	5.05929	5.05942	5.0621
		4	5.70566	5.70658	5.70531	5.70562	5.7077
		7	6.24371	6.24484	6.24339	6.24363	6.2438
0.4	35	1	0.24672	0.24557	0.24655	0.24655	0.2466
		4	1.34614	1.35059	1.34580	1.34600	1.3465
		7	2.15499	2.16031	2.15468	2.15485	2.1555
	40	1	1.76844	1.76612	1.76723	1.76657	1.7686
		4	3.38757	3.38370	3.38705	3.38735	3.3886
		7	4.35275	4.34809	4.35242	4.35266	4.3539
	45	1	5.28697	5.28768	5.28574	5.28659	5.2885
		4	6.51002	6.51040	6.50933	6.50982	6.5120
		7	7.38315	7.38983	7.38266	7.38299	7.3846
RMSE				$2.6343e{-3}$	$5.2876e{-4}$	4.5864e−4	$1.3235e{-3}$
CPU time				0.18	0.06	0.04	0.31

Table 3.18: Approximate option prices

σ	K	T	BIN1	BIN2	FEM1	FEM2	E23F
0.2	35	1	-0.0080	-0.0081	-0.0080	-0.0080	-0.0080
		4	-0.0901	-0.0911	-0.0900	-0.0901	-0.0898
		7	-0.1338	-0.1355	-0.1338	-0.1338	-0.1335
	40	1	-0.4693	-0.4698	-0.4691	-0.4692	-0.4703
		4	-0.4435	-0.4444	-0.4435	-0.4435	-0.4442
		7	-0.4287	-0.4299	-0.4287	-0.4287	-0.4280
	45	1	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
		4	-0.8812	-0.8803	-0.8794	-0.8819	-0.8878
		7	-0.7948	-0.7944	-0.7930	-0.7955	-0.8002
0.3	35	1	-0.0516	-0.0526	-0.0516	-0.0516	-0.0515
		4	-0.1741	-0.1758	-0.1741	-0.1741	-0.1738
		7	-0.2126	-0.2144	-0.2126	-0.2126	-0.2125
	40	1	-0.4695	-0.4699	-0.4693	-0.4691	-0.4699
		4	-0.4420	-0.4429	-0.4420	-0.4420	-0.4429
		7	-0.4256	-0.4268	-0.4256	-0.4256	-0.4262
	45	1	-0.9233	-0.9220	-0.9223	-0.9239	-0.9261
		4	-0.7266	-0.7260	-0.7253	-0.7271	-0.7248
		7	-0.6520	-0.6520	-0.6508	-0.6525	-0.6490
0.4	35	1	-0.1062	-0.1073	-0.1062	-0.1062	-0.1062
		4	-0.2260	-0.2277	-0.2259	-0.2259	-0.2257
		7	-0.2539	-0.2557	-0.2539	-0.2539	-0.2539
	40	1	-0.4668	-0.4673	-0.4666	-0.4663	-0.4670
		4	-0.4360	-0.4370	-0.4360	-0.4360	-0.4368
		7	-0.4173	-0.4186	-0.4173	-0.4173	-0.4182
	45	1	-0.8363	-0.8350	-0.8348	-0.8364	-0.8357
		4	-0.6475	-0.6474	-0.6463	-0.6476	-0.6468
		7	-0.5819	-0.5819	-0.5808	-0.5819	-0.5806
RMSE				$1.0593e{-}3$	$7.4351e{-}4$	2.9730e−4	$1.9262e{-}3$
CPU time				0.18	0.23	0.23	0.04

Table 3.19: Approximate hedge ratios

Method	RMSE ratio	CPU ratio
Finite difference–200	15.56	5.50
Accelerated binomial–150	13.38	0.67
Quadratic approximation	4.99	0.08
Recursive four-point	2.54	0.11
Compound option approx.	2.02	–
Binomial methods–150	1.0	1.0
Eq(23)/four-point	0.87	0.06
Eq(23)/integration	0.50	1.71
Euler/extrapolation–70	0.20	0.33
CN–140	0.17	0.22

Table 3.20: RMSE and CPU time ratios for put values

Method	RMSE ratio	CPU ratio
Finite difference–200	39.85	5.50
Recursive four-point	4.88	0.04
Euler/extrapolation–70	3.09	0.44
Compound option approx.	2.39	–
CN/extrapolation–70	2.09	0.44
Eq(23)/four-point	1.82	0.02
Binomial–150	1.00	1.00
Euler/extrapolation–140	0.70	1.28
CN/extrapolation–140	0.28	1.28

Table 3.21: RMSE and CPU time ratios for hedge ratios

Example 3.3. In this example, we examine the approximate early exercise prices. Consider a put with exercise price $K = \$40$, volatility $\sigma = 0.3$, interest rate $r = 0.0488$, and dividend rate $d = 0.0$. For the five meshes in Figure 3.3, our code for the Crank-Nicholson scheme took 0.01, 0.02, 0.13, 0.45 and 11.53 seconds. Figure 3.3 also shows that our method converges vary rapidly.

Recall that $S^*(t)$ is the largest value of S satisfying

$$K - S = V(S, t).$$

The bisection method (BS) can be used to find the approximation of $S^*(t)$ where $V(S,t)$ is computed by the binomial method. In Figure 3.4 we plot the approximate early exercise prices obtained by this method with 1000 and 10000 time steps for the binomial method (BS: $M = 1000$ and BS: $M = 10000$) and by the Crank-Nicholson scheme of our method with 1000 time steps (CN: $M = 1000$). We observe that the approximate early exercise prices given by the bisection method approach ours as the number of time steps for the binomial method increases. This implies that our method gives very accurate results. Compared with Figure 1 in [25], the early exercise prices obtained by our method are very good.

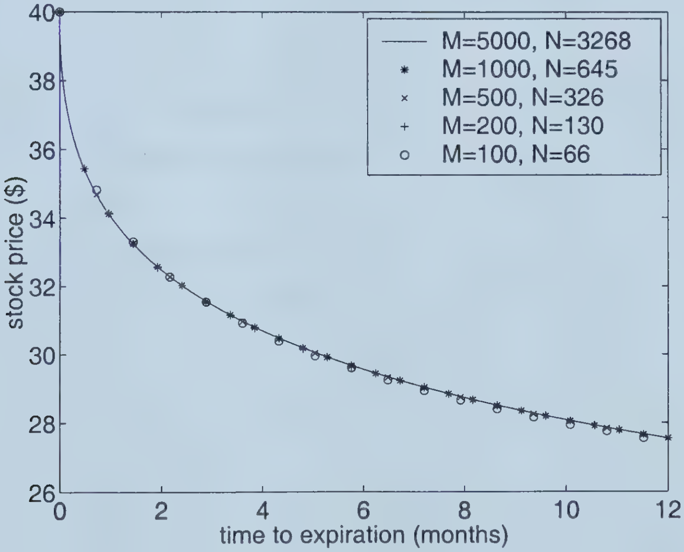

Figure 3.3: Approximate early exercise prices

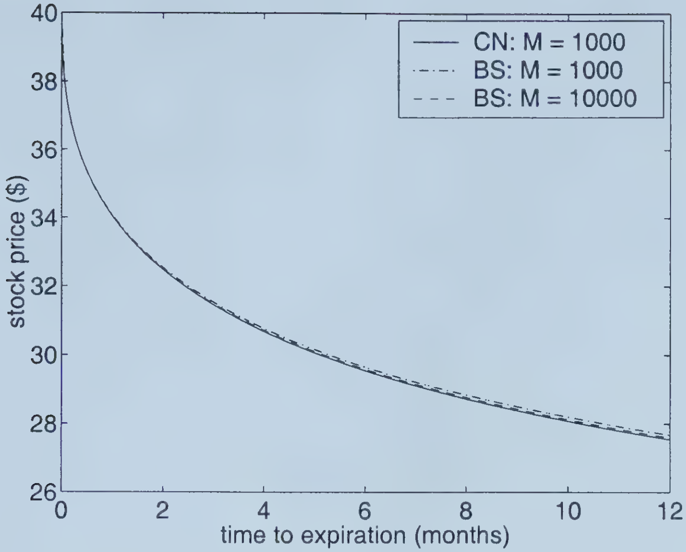

Figure 3.4: Comparison of approximate early exercise prices

Example 3.4. In this example, we shal examine the stability and efficiency of our method with the Crank-Nicholson scheme by varying the parameters σ, d and r. To this end, consider American put options on a commodity with exercise price $K = \$50$ and the expiration $T = 12$ months. Other parameters are as follows: volatility $\sigma = 0.2$, 0.4 or 0.6, interest $r = 0.05$, 0.1 or 0.15 and cost of carrying the commodity $b = -0.1$ or 0.05. As discussed in [15], we can apply the Black-Scholes model to these options by letting $d = r - b$. As in Example 3.2, we use option prices computed by the binomial method with 10000 time steps as "exact" in this example and next one.

We took M, the number of time steps, to be 120. As mentioned before, N, the number of steps in x, is chosen so that the mesh sizes in time and in x are almost equal. Thus N varies according to the different parameters chosen (see Table 3.22). The lower bound S_∞ of the early exercise price $S^*(t)$ and $X = \ln(K/S_\infty)$ are also listed in Table 3.22. We observe that the region $[0, T] \times [0, X]$ on which (3.42)–(3.46) is solved is very narrow in most cases. It should be pointed out that X is independent of the exercise price K. CPU times in seconds given in Table 3.22 (CUP1) include the time to compute

approximate option prices at $P = N/2 + Q + 1$ points when time to the expiration is 3 months and 12 months, where

$$Q = \begin{cases} N/2, & \text{if } N \le 100; \\ N/(2+N/50), & \text{if } N > 100, \end{cases}$$

and $/$ is understood as integer division. See Table3.22 for the values of Q and P. $N/2+1$ points in $[S\infty, K]$ are given by $K\ \exp(-2Xj/N)$ $(j = 0, 1, 2, \ldots, N/2)$ and Q points in (K, ∞) are given by $K\ \exp(2Xj/N)$ $(j = 1, 2, \ldots, Q)$. The maximum error (ME) and the root of mean squares errors (RMSE) are presented in Table 3.23–3.26, the number of total points are $N/2 + Q + 1$ for computing ME and RMSE at each time level. The results show that our method achieves almost the same approximations for the different parameters.

Finally, we consider examples of early exercise boundaries. CPU times in seconds to compute early exercise boundaries with 1000 time steps are given in the last column (CPU2) of Table 3.22. First, we fix the interest rate and the cost of carrying and vary the volatility (see Figure 3.5–3.7); second, we fix the volatility and the cost of carrying and vary the interest rate (see Figure 3.8–3.10). Figure 3.5–3.7 shows that the early exercise prices decrease as volatility increases, while interest rate and cost of carrying are held constant. Figure 3.8–3.10 shows that if volatility and cost of carrying are held constant, the early exercise prices increase as interest rate increases. These are as intuitively expected.

b	σ	r	S_∞	X	N	Q	P	CPU1	CPU2
-0.1	0.2	0.05	14.0589	1.26877	152	38	115	0.12	0.80
		0.10	21.2917	0.85371	104	26	79	0.09	0.56
		0.15	25.7519	0.66351	80	40	81	0.12	0.46
	0.4	0.05	10.0000	1.60944	192	48	145	0.15	0.99
		0.10	15.7786	1.15337	140	35	106	0.12	0.74
		0.15	19.6451	0.93420	112	28	85	0.05	0.62
	0.6	0.05	6.96155	1.97162	236	39	158	0.14	1.20
		0.10	11.5232	1.46766	176	44	133	0.15	0.96
		0.15	14.8259	1.21565	144	36	109	0.12	0.77
0.05	0.2	0.05	35.7143	0.33647	40	20	41	0.06	0.25
		0.10	37.8301	0.27892	32	16	33	0.05	0.22
		0.15	39.1055	0.24576	28	14	29	0.04	0.20
	0.4	0.05	19.2308	0.95551	116	29	88	0.09	0.66
		0.10	24.3082	0.72121	88	44	89	0.13	0.50
		0.15	27.2166	0.60819	72	36	73	0.11	0.42
	0.6	0.05	10.8696	1.52606	194	46	139	0.15	1.03
		0.10	15.9195	1.14448	136	34	103	0.11	0.79
		0.15	19.1470	0.95988	116	29	88	0.10	0.66

Table 3.22: S_∞, X, N, Q, P and CPU times in seconds

σ	r	ME	RMSE
0.2	0.05	$2.8551e{-4}$	$8.5075e{-5}$
	0.10	$2.5498e{-4}$	$9.2534e{-5}$
	0.15	$2.3114e{-4}$	$8.5846e{-5}$
0.4	0.05	$1.8857e{-3}$	$1.8167e{-4}$
	0.10	$1.8510e{-3}$	$2.0887e{-4}$
	0.15	$1.8335e{-3}$	$2.3279e{-4}$
0.6	0.05	$2.0823e{-3}$	$2.3751e{-4}$
	0.10	$2.0468e{-3}$	$2.5707e{-4}$
	0.15	$2.0822e{-3}$	$2.8583e{-4}$

Table 3.23: Approximation accuracy: $t = 3$ months, $b = -0.1$

σ	r	ME	RMSE
0.2	0.05	$1.0497e{-4}$	$3.9083e{-5}$
	0.10	$1.3577e{-4}$	$4.1342e{-5}$
	0.15	$1.4651e{-4}$	$4.7881e{-5}$
0.4	0.05	$2.0376e{-4}$	$7.0268e{-5}$
	0.10	$1.8657e{-4}$	$7.7078e{-5}$
	0.15	$1.6266e{-4}$	$7.8756e{-5}$
0.6	0.05	$5.7930e{-4}$	$1.1476e{-4}$
	0.10	$5.5163e{-4}$	$1.2154e{-4}$
	0.15	$5.1638e{-4}$	$1.1960e{-4}$

Table 3.24: Approximation accuracy: $t = 12$ months, $b = -0.1$

σ	r	ME	RMSE
0.2	0.05	$6.0122e{-4}$	$2.9497e{-4}$
	0.10	$8.3326e{-4}$	$3.5005e{-4}$
	0.15	$5.7825e{-4}$	$3.2276e{-4}$
0.4	0.05	$2.6819e{-3}$	$4.8736e{-4}$
	0.10	$2.6429e{-3}$	$4.7534e{-4}$
	0.15	$2.6025e{-3}$	$5.2320e{-4}$
0.6	0.05	$3.1115e{-3}$	$6.0263e{-4}$
	0.10	$3.1432e{-3}$	$6.7722e{-4}$
	0.15	$2.9999e{-3}$	$7.0499e{-4}$

Table 3.25: Approximation accuracy: $t = 3$ months, $b = 0.05$

σ	r	ME	RMSE
0.2	0.05	$3.5548e{-4}$	$2.0390e{-4}$
	0.10	$4.9513e{-4}$	$2.7377e{-4}$
	0.15	$1.1730e{-3}$	$3.8113e{-4}$
0.4	0.05	$4.7673e{-4}$	$2.5275e{-4}$
	0.10	$4.3653e{-4}$	$2.3639e{-4}$
	0.15	$4.2244e{-4}$	$2.4347e{-4}$
0.6	0.05	$1.0514e{-3}$	$3.5014e{-4}$
	0.10	$9.5729e{-4}$	$3.5088e{-4}$
	0.15	$8.9547e{-4}$	$3.3178e{-4}$

Table 3.26: Approximation accuracy: $t = 12$ months, $b = 0.05$

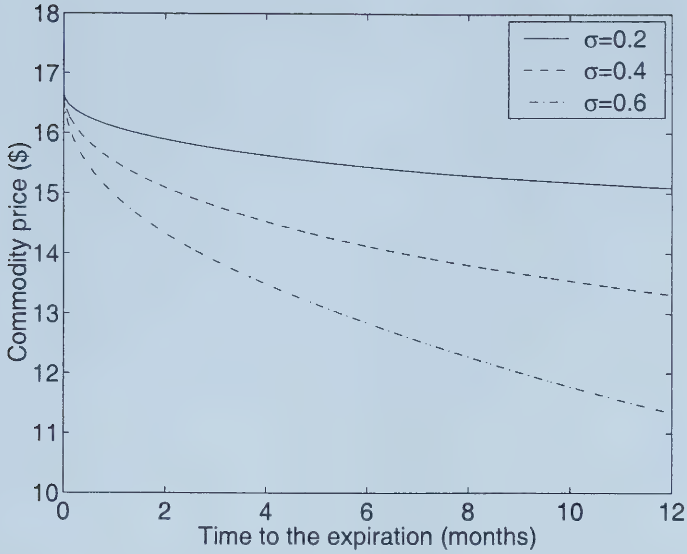

Figure 3.5: Early exercise prices: $K = \$50$, $r = 0.05$, $b = -0.1$

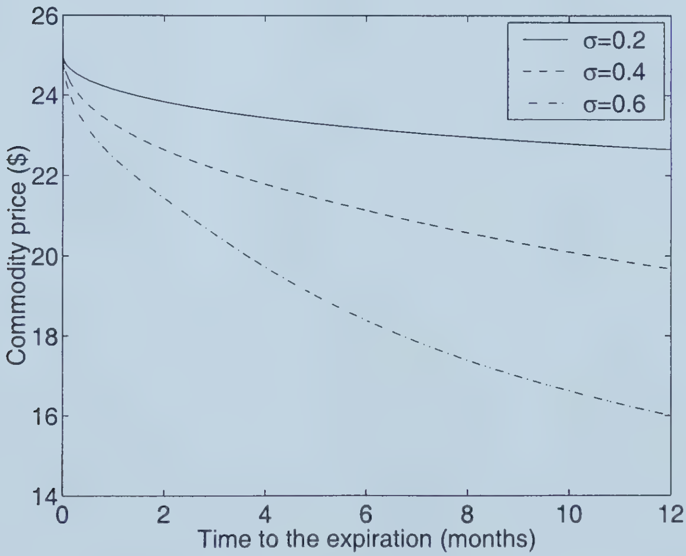

Figure 3.6: Early exercise prices: $K = \$50$, $r = 0.1$, $b = -0.1$

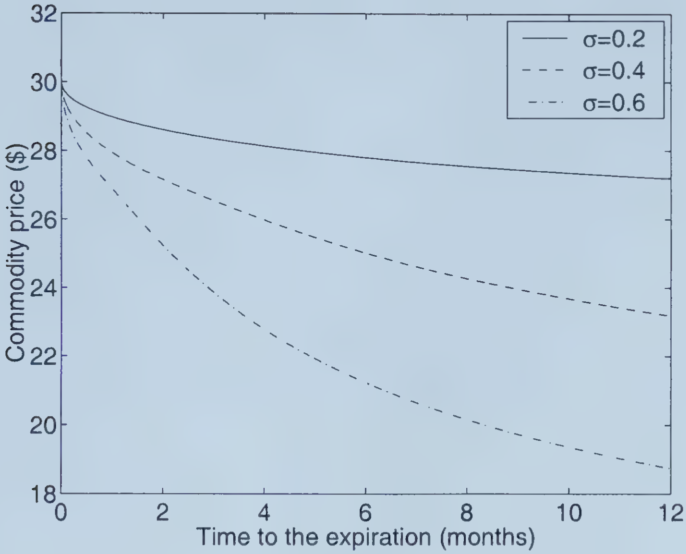

Figure 3.7: Early exercise prices: $K = \$50$, $r = 0.15$, $b = -0.1$

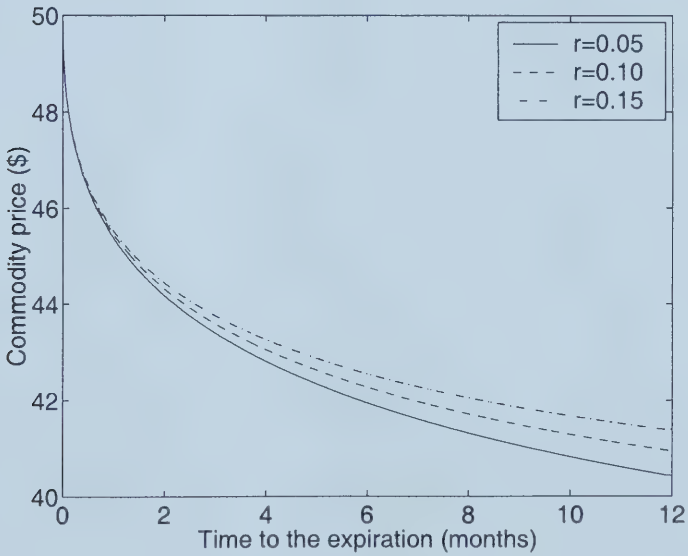

Figure 3.8: Early exercise prices: $K = \$50$, $\sigma = 0.2$, $b = 0.05$

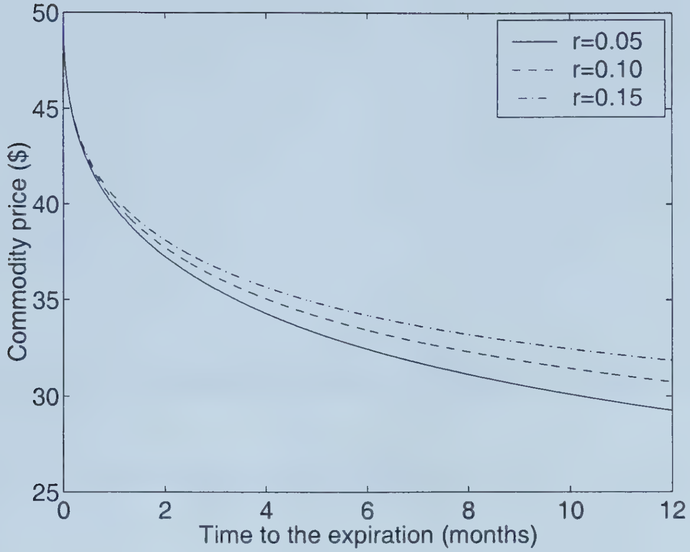

Figure 3.9: Early exercise prices: $K = \$50$, $\sigma = 0.4$, $b = 0.05$

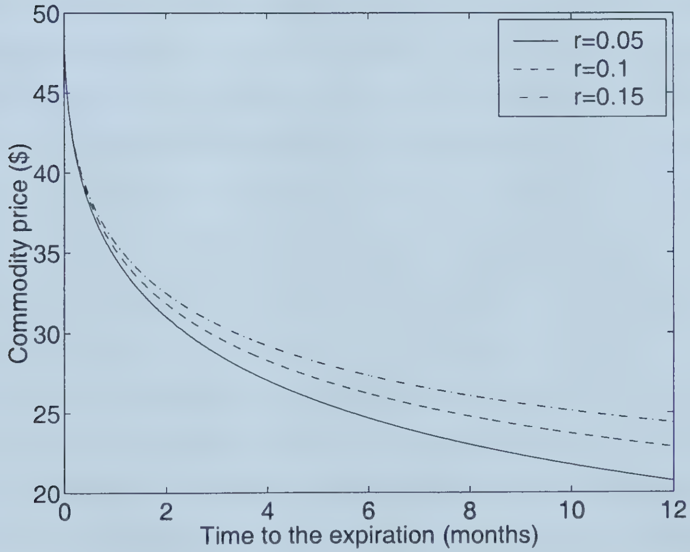

Figure 3.10: Early exercise prices: $K = \$50$, $\sigma = 0.6$, $b = 0.05$

Chapter 4

Pricing American Put Options on Zero-coupon Bonds

In this chapter we shall propose two types of numerical methods for pricing American put options on zero-coupon bonds under the CIR model: finite volume methods and finite element methods. The CIR model developed by Cox, Ingersoll and Ross in 1985 ([31]) is one of the most widely used term structure models and has several desirable features. In §4.1 we review pricing a zero-coupon bond under the CIR model. In §4.2 we introduce the free boundary problem for pricing American put options on zero-coupon bonds under the CIR model. In §4.3 and §4.4, we study the weak solution of the free boundary problem. The existence and uniqueness of the weak solution are proved. In §4.5 we construct our finite volume methods and finite element methods. The stability and convergence analysis is given in §4.5.3. In §4.6 we derive an error checking equation. Numerical examples are presented in §4.7 to examine our methods and to compare them with the simplified binomial method in [67, 68].

4.1 Zero-coupon Bonds

A bond is a contract, paid up front, which guarantees the holder a known amount on a known date in the future. The known amount is referred as the principal value or face value and the known date as the maturity date. The bond may also pay a known cash dividend (called the coupon) at fixed times during the life of the contract. If there is no dividend payment the bond is known as a zero-coupon bond, also known as a pure discount bond. The main purpose of a bond issue is the raising of capital, and the up-front premium can be thought of as a loan to the government or to the company that has issued the bonds.

Bonds and their options are interest rate derivatives. In all interest rate models, the short rate of interest $r(t)$ is considered to be a random process and governed by a stochastic differential equation. For the CIR model, $r(t)$ is assumed to follow the square-root process with the market price of risk $\lambda\sqrt{r(t)}/\sigma$, i.e.,

$$dr(t) = (\kappa r_\infty - (\kappa + \lambda)r(t))dt + \sigma\sqrt{r(t)}dW(t),$$

where $W(t)$ is a standard Brownian motion under the martingale measure Q, r_∞ is the long-term value of interest rate, κ is the speed of adjustment, and σ is a positive constant, and the market risk parameter λ is a constant. If λ is positive, zero, or negative, investors are risk averse, risk neutral, or risk seeking, respectively (see Chapter 9 of [65]).

Denote by $B(r, t)$ the value of the zero-coupon bond with face value $\$E$ and maturity date T^* when $r(t) = r$ at time t. Then $B(r, t)$ has the stochastic representation

$$B(r,t) = E^Q_{r,t}\left[\exp\left(-\int_t^{T^*} r(s)ds\right)\bigg|\mathcal{F}_t\right], \tag{4.1}$$

where $\{\mathcal{F}_t\}$ is the filtration generated by $W(t)$ and $r(s)$ is the solution of the

following dynamics

$$dr(s) = (\kappa r_\infty - (\kappa + \lambda)r(s))dt + \sigma\sqrt{r(s)}dW(s), \quad s > t, \tag{4.2}$$

$$r(t) = r. \tag{4.3}$$

Since $e^{-\int_0^t r(s)ds}B(r(t), t)$ is a martingale under Q, it follows from Ito's lemma that $B(r, t)$ is the solution of the partial differential equation:

$$B_t + \frac{1}{2}\sigma^2 r B_{rr} + (\kappa r_\infty - (\kappa + \lambda)r)B_r - rB = 0, \quad 0 \le t < T^*, \ r > 0. \tag{4.4}$$

Furthermore $B(r, t)$ satisfies the following final and boundary conditions:

$$B(r, T^*) = E, \quad r \ge 0; \tag{4.5}$$

$$\lim_{r \to +\infty} B(r, t) = 0, \quad 0 \le t < T^*, \tag{4.6}$$

$$B(r, t) \text{ remains finite as } r \to 0^+. \ \ 0 \le t < T^*, \tag{4.7}$$

Evaluating the expectation (4.1) or solving the final-boundary value problem (4.4)–(4.7), one can get

$$B(r, t) = EA(t)e^{-C(t)r},$$

where

$$A(t) = \left(\frac{c_1 e^{c_2 s}}{c_2(e^{c_1 s} - 1) + c_1}\right)^{c_3}, \quad C(t) = \frac{e^{c_1 s} - 1}{c_2(e^{c_1 s} - 1) + c_1}, \quad s = T^* - t,$$
$$c_1 = (b^2 + 2\sigma^2)^{\frac{1}{2}}, \quad c_2 = (b + c_1)/2, \quad c_3 = 2a/\sigma^2,$$
$$a = \kappa r_\infty, \quad b = \kappa + \lambda.$$

It is easy to check that $A'(t) > 0$ and $C'(t) < 0$, i.e, $A(t)$ is strictly increasing and $C(t)$ is strictly decreasing. Therefore $B(r, t)$ is a increasing function of t and a decreasing function of r, which is as expected.

4.2 American Put Options

Consider the American put option with exercise price \$$K$ and expiry date T, which is written on a zero-coupon bond with face value \$$E$ and maturity

date T^* ($> T$). Recall that American contingent claims can be formulated as optimal stopping problems (see Bensoussan (1984) and Karatzas (1988, 1989)). Denote by $P(r,t)$ the price of the put for $r(t) = r$ at time t. Then

$$(4.8) \qquad P(r,t) = \sup_{\tau \in \mathcal{T}_{[t,T]}} E^Q_{r,t}\left[\exp\left(-\int_t^\tau r(s)ds\right) g(r(\tau),\tau) \middle| F_t\right],$$

where $r(s)$ is the solution of initial value problem (4.2)–(4.3), $\mathcal{T}_{[t,T]}$ is the set of all stopping time taking value in $[t,T]$, and

$$g(r,t) = \max(K - B(r,t), 0)$$

is the payoff of the put.

Similar to the American options on stocks, there is a critical interest rate $r^*(t)$ for any time t. It is the smallest value of the interest rate at which the exercise of the put becomes optimal. We may call $r^*(t)$ the early exercise interest rate. It is known that $P(r,t)$ and $r^*(t)$ are the solution of the following free boundary problem (see [29]):

$$(4.9) \qquad P_t + LP = 0, \quad P(r,t) > g(r,t), \quad 0 < r < r^*(t), \ 0 \le t < T,$$

$$(4.10) \qquad P(r,t) = g(r,t), \quad P_r(r,t) = g_r(r,t), \quad r \ge r^*(t), \ 0 \le t \le T,$$

$$(4.11) \qquad P(0,t) = g(0,t), \quad 0 \le t \le T,$$

$$(4.12) \qquad P(r,T) = g(r,T), \quad r \ge 0,$$

where

$$LP = \frac{1}{2}\sigma^2 r P_{rr} + (a - br)P_r - rP.$$

Remark 4.1. K should be strictly less than $B(0,T) = EA(T)$ which is the maximum of $B(r,t)$ on $[0,\infty) \times [0,T]$. Otherwise, exercise would be never optimal (see [29]). In fact, if $K \ge B(0,T)$, then

$$K \ge B(r,t), \ r \ge 0, \ 0 \le t \le T.$$

Hence it follows from (4.8) that

$$P(r,t) = K - B(r,t), \ r \ge 0, \ 0 \le t \le T.$$

Remark 4.2. Notice that κ, r_∞ and λ do not occur in $B(r,t)$ and coefficients of differential operator L separately, but only in a and b. The constants σ, a and b are the truly independent parameters of the CIR model.

Remark 4.3. For $t \in [0,T]$, let $\tilde{r}(t)$ be the solution to $B(r,t) = K$, i.e.,

$$\tilde{r}(t) = \frac{\log(EA(t)/K)}{C(t)}.$$

Then we have

(i) $B(r,t) < K$ if $r > \tilde{r}(t)$ and $B(r,t) > K$ if $r < \tilde{r}(t)$;

(ii) $\tilde{r}(t) \in C^\infty[0,T]$ and is a increasing function of t;

(iii) $r^*(t) > \tilde{r}(t)$, $\forall t \in [0,T)$ and $r^*(T) = \tilde{r}(T)$.

Conclusions (i) and (ii) follows from the definitions of $A(t)$ and $C(t)$. Notice that $P(r,t)$ is always positive for $t < T$. Conclusion (iii) follows from (4.9), (4.10) and (4.12).

Remark 4.4. For $t_1 < t_2$, if $\tau \in \mathcal{T}_{[t_2,T]}$ then $\tau \in \mathcal{T}_{[t_1,T]}$. This implies from (4.8) that $P(r,t)$ is a decreasing function of t. By using the maximum principle one can show from (4.9)–(4.12) that $P(r,t)$ is a increasing function of r. Although these properties are similar to the ones for an American put option on a stock in [46], we can not prove that $r^*(t)$ is a monotone function of t. Indeed, the numerical results in §4.5 show that $r^*(t)$ is a concave downward function.

4.3 Weak Solution

In this and next section we shall study the existence and uniqueness of the week solution of free boundary problem (4.9)–(4.12).

From now on we suppose that $r^*(t)$ has an upper bound R. Let

$$p(r,t) = P(r, T-t), \quad G(r,t) = g(r, T-t).$$

Then (4.9)–(4.12) become the following variational inequality:

$$p_t - Lp \geq 0, \; p \geq G, \; 0 < r < R, \; 0 < t \leq T, \tag{4.13}$$

$$(p_t - Lp)(p - G) = 0, \; 0 < r < R, \; 0 < t \leq T, \tag{4.14}$$

$$p(0,t) = G(0,t), \; 0 \leq t \leq T, \tag{4.15}$$

$$(4.16)\qquad p(R,t)=G(R,t),\ 0\le t\le T,$$

$$(4.17)\qquad p(r,0)=G(r,0),\ 0\le r\le R.$$

In order to remove that degenerate factor r in the highest order derivative of L, we consider the following transforms:

$$(4.18)\qquad x=\sqrt{r},\quad p(r,t)=x^{-\alpha}e^{\gamma t}u(x,t),$$

where α and γ are constants to be determined. We have by calculation

$$p_t-Lp=x^{-\alpha}\mathrm{e}^{\gamma t}\left(u_t-\frac{\sigma^2}{8}u_{xx}+c_1(x)u_x+c_2(x)u\right),$$

where

$$c_1(x)=\xi_1x^{-1}+\xi_2x,\quad c_2(x)=\xi_3x^{-2}+x^2+\gamma-\frac{\alpha b}{2},$$
$$\xi_1=\frac{\sigma^2}{8}\left(1+2\alpha-\frac{4a}{\sigma^2}\right),\quad \xi_2=\frac{b}{2},\quad \xi_3=\frac{\sigma^2\alpha}{8}\left(\frac{4a}{\sigma^2}-\alpha-2\right),$$

Hence (4.13)–(4.17) become

$$(4.19)\quad u_t-\frac{\sigma^2}{8}u_{xx}+c_1(x)u_x+c_2(x)u\ge 0,\quad u\ge F(x,t),\quad (x,t)\in\Omega_T,$$

$$(4.20)\quad \left(u_t-\frac{\sigma^2}{8}u_{xx}+c_1(x)u_x+c_2(x)u\right)(u-F(x,t))=0,\quad (x,t)\in\Omega_T,$$

$$(4.21)\quad u(0,t)=F(0,t),\quad 0\le t\le T,$$

$$(4.22)\quad u(X,t)=F(X,t),\quad 0\le t\le T,$$

$$(4.23)\quad u(x,0)=F(x,0),\quad 0\le x\le X,$$

where $\Omega_T=\Omega\times(0,T)$, $\Omega=(0,X)$, $X=\sqrt{R}$, and

$$F(x,t)=x^{\alpha}\mathrm{e}^{-\gamma t}G(x^2,t).$$

Write

$$W_F(0,T)=\{v:\ v\in L^2(0,T;H^1(\Omega)),v(0,t)=F(0,t)\text{ and }$$
$$v(X,t)=F(X,t)\text{ for a.e. }t\in[0,T],\ v_t\in L^2(0,T;H^{-1}(\Omega))\}.$$

and for $t\in[0,T]$

$$\Pi(t)=\{v:\ v\in H_0^1(\Omega),\ v(x)\ge F(x,t)\text{ for a.e. }x\in[0,X]\}.$$

Then the variational problem for (4.19)–(4.23) is: Find $u \in W_F(0,T)$ such that $u(x,0) = F(x,0)$ for a.e. $x \in [0,X]$ and for a.e. $t \in (0,T)$

$$(u_t, v-u) + a(u, v-u) \geq 0 \tag{4.24}$$

for all $v \in \Pi(t)$.

It is always easier to deal with homogeneous boundary conditions. Let $w = u - F$. Notice that

$$G(r,t) = g(r, T-t) = (K - B(r, T-t))H(r - \tilde{r}(T-t)),$$

where $H(z)$ is the Heaviside function defined by

$$H(z) = \begin{cases} 1, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

By some calculation, we can conclude that w is the solution of the variational problem: Find $w \in W(0,T)$ with $w(0) = 0$ such that for a.e. $t \in (0,T]$, $w(t) \in \Pi$ and

$$(w_t, v-w) + a(w, v-w) \geq f(t, v-w), \ \forall v \in \Pi \tag{4.25}$$

where

$$\begin{aligned}
&W(0,T) = \{v : \ v \in L^2(0,T; H_0^1(\Omega)), \ v_t \in L^2(0,T; H^{-1}(\Omega))\}, \\
&\Pi = \{v : \ v \in H_0^1(\Omega), \ v \geq 0\}, \\
&a(u,v) = \frac{\sigma^2}{8}(u_x, v_x) + (c_1 u_x + c_2 u, v), \\
&f(t,v) = -(F_t, v) - a(F, v) = -(\phi, v) + \psi(t) v\left(\tilde{x}(t)\right). \\
&\phi(x,t) = K x^{2+\alpha} \mathrm{e}^{-\gamma t} H(x - \tilde{x}(t)), \\
&\psi(t) = \frac{\sigma^2}{4} K C (T-t) \tilde{x}^{1+\alpha}(t) \mathrm{e}^{-\gamma t}, \\
&\tilde{x}(t) = \sqrt{\tilde{r}(T-t)}.
\end{aligned}$$

It is apparent that two variational problems (4.24) and (4.25) are equivalent. For a solution $w(x,t)$ of the variational problem (4.25), we shall call $P(r,t) = r^{-\frac{\alpha}{2}} \mathrm{e}^{\gamma t} w(\sqrt{r}, t) + g(r,t)$ a weak solution of the free boundary problem (4.9)–(4.12).

4.4 Uniqueness and Existence of the Weak Solution

In this section we shall show that the variational problem (4.25) has a unique solution. Hence the free boundary problem (4.9)–(4.12) has a unique weak solution.

Now we specify α and γ as follows:

$$\gamma = \frac{(1+2\alpha)b}{4}$$

and $\alpha \in [0, 1)$ is chosen in the following way::

(1) $\alpha \in [0,\ \rho]$, if $0 < \rho < 3/8$,

(2) $\alpha \in (4\rho - 3/2,\ 1/2)$, if $3/8 \leq \rho < 1/2$,

(3) $\alpha \in (1/2,\ 4\rho - 3/2)$, if $1/2 < \rho \leq 3/4$,

(4) $\alpha \in (1/2,\ 2\rho - 1/2 - \sqrt{(4\rho-1)(4\rho-3)}/2)$, if $\rho > 3/4$.

where $\rho = a/\sigma^2$.

Lemma 4.1. *For α and γ as in the above, if $a/\sigma^2 \neq 1/2$, then the bilinear form $a(\cdot,\cdot)$ is coercive in $H_0^1(\Omega)$, i.e., there is a positive constant γ_0 such that*

$$a(v, v) \geq \gamma_0 \|v_x\|_{0,\Omega}^2, \quad \forall v \in H_0^1(\Omega). \tag{4.26}$$

Proof. For $v \in H_0^1(\Omega)$, integration by parts gives

$$a(v, v) = \frac{\sigma^2}{8}\left(\|v_x\|_{0,\Omega}^2 + (-\nu + 2\alpha\nu - \alpha^2)\|x^{-1}v\|_{0,\Omega}^2\right) + \|xv\|_{0,\Omega}^2, \tag{4.27}$$

where $\nu = 2\rho - 1/2$. For Case (1), we have $\nu < 1/4$. Hence for $\alpha = 0$, we have from Lemma 2.1

$$a(v, v) \geq \frac{\sigma^2}{8}(1 - 4\nu)\|v_x\|_{0,\Omega}^2,$$

that is, (4.26) holds with $\gamma_0 = \sigma^2(1 - 4\nu)/8$.

Next we only show (4.26) in Case (4). The other cases can be proved in the same way. For $\rho > 3/4$, we have $\nu > 1$. For α satisfying

$$-\nu + 2\alpha\nu - \alpha^2 \leq 0, \tag{4.28}$$

by Lemma 2.1 again, we get from (4.27) that for $v \in H_0^1(\Omega)$

$$a(v,v) \geq \frac{\sigma^2}{8}(1+4(-\nu+2\alpha\nu-\alpha^2))\|v_x\|^2_{0,\Omega}.$$

We also require that α satisfies that

$$1+4(-\nu+2\alpha\nu-\alpha^2) > 0. \tag{4.29}$$

Case (4) follows from solving inequalities (4.28) and (4.29). □

Remark 4.5. For $a/\sigma^2 = 1/2$, by using $v^\epsilon(x)$ defined as in the proof of Lemma 2.1, we obtain

$$\lim_{\epsilon\to 0+} \frac{a(v^\epsilon, v_\epsilon)}{\|v_x^\epsilon\|^2_{0,\Omega}} = -\frac{\sigma^2}{8}(2\alpha-1)^2.$$

Hence, when $a/\sigma^2 = 1/2$, the bilinear form $a(\cdot,\cdot)$ is not coercive for any α. But we still have

$$a(v,v) \geq \|xv\|^2_{0,\Omega}, \quad \forall v \in H_0^1(\Omega).$$

Remark 4.6. We may consider the following variable transforms:

$$x = \sqrt{r}, \quad p(r,t) = x^{-\alpha}e^{\beta x^2+\gamma t}u(x,t)$$

with

$$\alpha = \frac{2a}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{b}{\sigma^2}, \quad \gamma = \frac{ab}{\sigma^2}.$$

Then we have a variational inequality problem for $u(x,t)$ with a coercive symmetric bilinear when $a/\sigma^2 \neq 1/2$. Although α may be larger than 1 in most cases, this formulation also gives very good approximations of option prices when $a/\sigma^2 \geq 1/4$.

Remark 4.7. In [8], we have considered the following transform:

$$v(r,t) = r^{-\alpha}e^{\gamma t}P(r, T-t).$$

For properly chosen α in $[0,1)$ and $\gamma > 0$, we also get a variational inequality problem for $v(r,t)$ with a coercive bilinear form. Numerical results in [8] show that the corresponding finite element method has poorer accuracy than others.

Theorem 4.1. *For $a/\sigma^2 \neq 1/2$ and α and γ are chosen as in Lemma 4.1, (4.25) has a unique solution $w(x,t)$ in $W(0,T)$.*

Proof. Since the formulation of variational inequality (4.25) is similar to that in [12, 13], we only outline the proof of the theorem.

We first show the uniqueness of the solution $w(x,t)$. Suppose that w_1 and w_2 are two solutions of (4.25). Let $e = w_1 - w_2$. Then we have

$$(e_t, e) + a(e, e) \leq 0.$$

Thus,

$$\frac{1}{2}(e^2(T) - e^2(0)) + \int_0^T a(e(t), e(t))dt \leq 0.$$

This together with $e(0) = 0$ implies that $e = 0$ a.e. on $\Omega \times (0,T)$. Hence, $w_1 = w_2$, that is, the solution of (4.25) is unique.

Next we prove the existence. For a positive integer n, define

$$H_n(z) = \begin{cases} 1, & x \geq \frac{1}{n}, \\ nx, & 0 < x < \frac{1}{n}, \\ 0, & x \leq 0. \end{cases}$$

We claim that the following nonlinear problem has a unique solution $w^n \in W(0,T)$ with $w^n(0) = 0$:

$$(w_t^n, v) + a(w^n, v) + (\phi H_n(w^n), v) = \psi(t)v(\tilde{x}(t)), \quad \forall v \in H_0^1(\Omega). \tag{4.30}$$

Furthermore, $\{w^n\}$ is a bounded nonnegative function sequence in $W(0,T)$. The uniqueness can be proved in the same way as above. The existence and boundedness of w^n follows from the Schauder fixed point theorem (Chapter 5 of [45]). Let

$$p = \begin{cases} 0, & w^n \geq 0, \\ v, & w^n < 0. \end{cases}$$

Then $p \in H_0^1(\Omega)$ and

$$w_t^n p = p_t p, \quad w_x^n p_x = (p_x)^2, \quad w_x^n p = p_x p, \quad w^n p = p^2, \quad H_n(w^n)p = 0.$$

Hence, for $v = p$ in 4.30, we get

$$(p_t, p) + a(p, p) = \psi(t)p(\tilde{x}(t)) \leq 0.$$

Thus, $p = 0$, that is, w^n is nonnegative.

Recall that a bounded sequence in a Hilbert space has a weakly convergent subsequence and a bounded sequence in a Banach space has a weakly* convergent subsequence. We may assume that $\{w^n\}$ weakly converges to a nonnegative function w in $W(0,T)$ and that $\{H_n(w^n)\}$ weakly* converges to G in $L^\infty((\Omega) \times (0,T))$. It is clear that $0 \leq G \leq 1$. It follows from Lemma 5.1 of [13] that

$$G(x,t) = 1, \quad \text{if } u(x,t) > 0. \tag{4.31}$$

Then by letting $n \to \infty$ in (4.30), we have

$$(w_t, v) + a(w, v) + (\phi G, v) = \psi(t)(\delta(x - \tilde{x}(t)), v), \quad \forall v \in H_0^1(\Omega),$$

i.e., for $v \in H_0^1(\Omega)$,

$$(w_t, v - w) + a(w, v - w) + (\phi G, v - w) = \psi(t)(\delta(x - \tilde{x}(t)), v - w).$$

Notice that ϕ is a nonnegative function and that from (4.31)

$$G(v - w) \leq v - w, \quad \forall v \in \Pi.$$

We have

$$(w_t, v - w) + a(w, v - w) + (\phi, v - w) \geq \psi(t)(\delta(x - \tilde{x}(t)), v - w), \quad \forall v \in \Pi.$$

Therefore, (4.25) has a solution w in $W(0,T)$. □

4.5 Numerical Solutions

In this section we construct finite volume methods for (4.13)–(4.17) and finite element methods for (4.24) and (4.25).

4.5.1 Finite volume approximations

Finite volume method is also called box method or generalized difference method by other authors. We refer to [55] for theoretical analysis and applications.

It is easy to rewrite (4.13)–(4.17) in the following symmetric form (see [14, 73]):

$$w_1 p_t - (w_2 p_r)_r + w_3 p \geq 0, \quad p \geq G, \quad 0 < r < R, \ 0 < t \leq T, \tag{4.32}$$

$$(w_1 p_t - (w_2 p_r)_r + w_3 p)(p - G) = 0, \quad 0 < r < R, \ 0 < t \leq T, \tag{4.33}$$

$$p(0,t) = G(0,t), \quad 0 \leq t \leq T, \tag{4.34}$$

$$p(R,t) = G(R,t), \quad 0 \leq t \leq T, \tag{4.35}$$

$$p(r,0) = G(r,0), \quad 0 \leq r \leq R, \tag{4.36}$$

where

$$w_1(r) = r^{\nu-1} e^{-\delta r}, \ w_2(r) = \frac{\sigma^2}{2} r^{\nu} e^{-\delta r}, \ w_3(r) = r^{\nu} e^{-\delta r}, \ \nu = \frac{2a}{\sigma^2}, \ \delta = \frac{2b}{\sigma^2}.$$

Let $0 = r_0 < r_1 < \ldots < r_N = R$ be a regular partition of $\Omega = [0, R]$, i.e., there is a positive constant $\varrho \in (0,1)$ independent of h such that

$$\varrho h \leq r_j - r_{j-1} \leq h, \quad j = 1, 2, \ldots, N, \tag{4.37}$$

where N is a positive integer and $h = \max_{1 \leq j \leq N}(r_j - r_{j-1})$. For another positive integer M, let $\tau = T/M$ be the step size in t and $t_m = m\tau$, $m = 0, 1, \ldots, M$. First, we use the backward difference quotient to approximate $p_t(r, t_m)$, i.e.,

$$p_t(r, t_m) \approx \delta_\tau p(r, t_m) = \frac{p(r, t_m) - p(r, t_{m-1})}{\tau}.$$

Next we discretize the spatial variable by the finite volume method. For $j = 1, 2, \ldots, N$, integrating

$$w_1(r)\delta_\tau p(r, t_m) - (w_2(r) p_r(r, t_m))_r + w_3(r) p(r, t_m)$$

in r over $\left[r_{j-\frac{1}{2}}, r_{j+\frac{1}{2}}\right] = \left[\dfrac{r_j + r_{j+1}}{2}, \dfrac{r_j + r_{j-1}}{2}\right]$ and using the following approximations

$$p_r\left(r_{j\pm\frac{1}{2}}, t_m\right) \approx \frac{p(r_j, t_m) - p(r_{j\pm 1}, t_m)}{r_j - r_{j\pm 1}},$$

$$\int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} w_i(r)p(r, t_k)dr \approx \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} w_i(r)dr p(r_j, t_k), \quad i = 1, 2, 3, \quad k = m-1, m,$$

we get an approximation of the problem (4.32)–(4.36) as follows: for $m = 1, 2, \ldots, M$,

$$\begin{cases} D\delta_\tau U^m + AU^m \geq F^m, \quad U^m \geq H^m, \\ (D\delta_\tau U^m + AU^m - F^m, U^m - H^m) = 0, \end{cases}$$

where $(\cdot, \cdot)$ denotes the inner product on R^{N-1}, and

$$A = \begin{pmatrix} a_1 & -b_1 & & & & \\ -b_1 & a_2 & -b_2 & & & \\ & -b_2 & a_3 & -b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b_{N-3} & a_{N-2} & -b_{N-2} \\ & & & & -b_{N-2} & a_{N-1} \end{pmatrix}, \quad U^m = \begin{pmatrix} p_1^m \\ p_2^m \\ \vdots \\ p_{N-1}^m \end{pmatrix},$$

$$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_{N-1} \end{pmatrix}, \quad F^m = \begin{pmatrix} f_1^m \\ f_2^m \\ \vdots \\ f_{N-1}^m \end{pmatrix}, \quad H^m = \begin{pmatrix} g_1^m \\ g_2^m \\ \vdots \\ g_{N-1}^m \end{pmatrix},$$

p_j^m is the approximation of $p(r_j, t_m)$ for $j = 1, 2, \ldots, N-1$, and

$$d_j = \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} w_1(r)dr, \quad j = 1, 2, \ldots, N-1,$$

$$b_j = \frac{w_2\left(r_{j+\frac{1}{2}}\right)}{r_{j+1} - r_j}, \quad j = 0, 1, \ldots, N-1,$$

$$c_j = \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} w_3(r)dr, \quad j = 1, 2, \ldots, N-1,$$

$$a_j = b_{j-1} + b_j + c_j, \quad j = 1, 2, \ldots, N-1,$$

$$g_j^m = G(r_j, t_m), \quad j = 0, 1, \ldots, N,$$
$$f_1^m = b_0 g_0^m, \; f_{N-1}^m = b_{N-1} g_N^m, \; f_j^m = 0, \; j = 2, 3, \ldots, N-2.$$

Furthermore, we have the following general approximate schemes:

$$\begin{cases} D\delta_\tau U^m + AU^{m-\theta} \geq F^{m-\theta}, \quad U^m \geq H^m, \\ \left(D\delta_\tau U^m + AU^{m-\theta} - F^{m-\theta}, U^m - H^m\right) = 0, \end{cases} \tag{4.38}$$

for $m = 1, 2, \ldots, M$, where θ is a given number in $[0, 1]$ and

$$U^{m-\theta} = \theta U^{m-1} + (1-\theta)U^m, \quad F^{m-\theta} = \theta F^{m-1} + (1-\theta)F^m.$$

For $\theta = 0, \; 0.5, \; 1$, (4.38) is the backward Euler scheme, the Crank–Nicholson scheme, and the forward Euler scheme, respectively.

4.5.2 Finite element approximations

For a regular partition of $[0, X]$ (see §3.3): $0 = x_0 < x_1 < \cdots < x_N = X$, let $\phi_i(x)$ $(i = 0, 1, 2 \ldots, N)$ be the basis functions of linear finite element space, i. e.,

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0}, & x \in [x_0, x_1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}], \quad i = 1, 2, \ldots, N-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_N(x) = \begin{cases} \dfrac{x - x_{N-1}}{x_N - x_{N-1}}, & x \in [x_{N-1}, x_N], \\ 0, & \text{otherwise}. \end{cases}$$

We first consider finite element approximations of $u(x, t)$. Write

$$\Pi_h^m = \Bigg\{ v(x) = \sum_{i=0}^{N} \xi_i \phi_i(x) : \; \xi_0 = F(0, t_m), \; \xi_N = F(X, t_m),$$
$$\xi_i \geq F(x_i, t_m), \; i = 1, 2, \ldots, N-1 \Bigg\},$$

for $m = 1, 2, \ldots, M$, where $h = \max_{1 \le j \le N}(x_j - x_{j-1})$. Then the finite element approximation of (4.24) is: Find $u_h^m \in \Pi_h^m$ such that

$$(4.39) \qquad (\delta_\tau u_h^m, v - u_h^m) + a(u_h^{m-\theta}, v - u_h^m) \ge 0,$$

for all $v \in \Pi_h^m$, where θ is a given number in $[0, 1]$, $u_h^{m-\theta}$ is defined in the same way as $U^{m-\theta}$, and u_h^0 is the interpolation of $u(x, 0) = F(x, 0)$, i.e.,

$$u_h^0 = \sum_{i=0}^{N} F(x_i, 0)\phi_i(x).$$

Next consider finite element approximations of $w(x, t)$. Write

$$\Pi_h = \left\{ v(x) = \sum_{i=1}^{N-1} \xi_i \phi_i(x) : \ \xi_i \ge 0, \ i = 1, 2, \ldots, N-1 \right\},$$

It is apparent that Π_h is subset of Π. Also, as h goes to 0^+, Π_h converges to Π. The finite element approximations of (4.25) is: Find $w_h^m \in \Pi_h$ such that

$$(4.40) \qquad (\delta_\tau w_h^m, v - w_h^m) + a(w_h^{m-\theta}, v - w_h^m) \ge f(t, v - w_h^m)$$

for all $v \in \Pi_h$, where θ is a given number in $[0, 1]$, $w_h^{m-\theta}$ is defined in the same way as $U^{m-\theta}$, and $w_h^0 = 0$.

For $\theta = 0, \ 0.5, \ 1$, (4.39) and (4.40) are the backward Euler scheme, the Crank–Nicholson scheme, and the forward Euler scheme, respectively. Inequality (4.39) and (4.40) can also be written as matrix forms similar to (4.38). Here D is a symmetric tridiagonal matrix and A is a non-symmetric positive definite tridiagonal matrix.

4.5.3 Stability and convergence

We first study the stability and convergence of the finite volume method (4.38). To this end, we need the following discrete weighted norms:

$$\|V\|_{0,w_1,h}^2 = \sum_{j=1}^{N-1} d_j v_j^2, \quad \|V\|_{1,w_2,h}^2 = \sum_{j=1}^{N} b_{j-1}\left(v_j - v_{j-1}\right)^2 + \sum_{j=1}^{N-1} c_j v_j^2$$

for $V = (v_0, v_1, v_2, \ldots, v_N)^T \in R^{N+1}$. Let $R_0^{N+1} = \left\{V \in R^{N+1} : v_0 = v_N = 0\right\}$. Then R_0^{N+1} is a subspace of R^{N+1}, just as $H_0^1(0, R)$ is a subspace of $H^1(0, R)$. It is easy to check that

$$\|V\|_{0,w_1,h}^2 = \left(D\widetilde{V}, \widetilde{V}\right), \quad \forall V \in R^{N+1}, \tag{4.41}$$

$$\|V\|_{1,w_2,h}^2 = \left(A\widetilde{V}, \widetilde{V}\right), \quad \forall V \in R^{N+1}, \tag{4.42}$$

where $\widetilde{V} = (v_1, v_2, \ldots, v_{N-1})$. As usual, we need an inverse estimate for the above norms. This is given in the following lemma.

Lemma 4.2. *There is a positive constant Λ independent of h and N such that*

$$\|V\|_{1,w_2,h}^2 \le \Lambda h^{-2} \|V\|_{0,w_1,h}^2 \tag{4.43}$$

for all $V \in R_0^{N+1}$.

Proof. Simple calculation gives

$$\begin{aligned}
\|V\|_{1,w_2,h}^2 &= \sum_{j=1}^{N} w_2\left(r_{j-\frac{1}{2}}\right) \frac{\left(v_j - v_{j-1}\right)^2}{r_j - r_{j-1}} + \sum_{j=1}^{N-1} \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} w_3(r) dr v_j^2 \\
&\le \frac{2}{\varrho h} \sum_{j=1}^{N} w_2\left(r_{j-\frac{1}{2}}\right) \left(v_j^2 + v_{j-1}^2\right) + \sum_{j=1}^{N-1} \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} w_3(r) dr v_j^2 \\
&= \frac{1}{h} \sum_{j=1}^{N-1} \left[\frac{2}{\varrho}\left(w_2\left(r_{j-\frac{1}{2}}\right) + w_2\left(r_{j+\frac{1}{2}}\right)\right) + h \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} w_3(r) dr\right] v_j^2.
\end{aligned}$$

Notice that

$$\begin{aligned}
&w_2\left(r_{j-\frac{1}{2}}\right) + w_2\left(r_{j+\frac{1}{2}}\right) = \frac{\sigma^2}{2}\left(r_{j-\frac{1}{2}}^{\nu} e^{-\delta r_{j-\frac{1}{2}}} + r_{j+\frac{1}{2}}^{\nu} e^{-\delta r_{j+\frac{1}{2}}}\right) \\
&= \frac{\sigma^2}{2\left(r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}\right)} \left[e^{\delta\left(r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}\right)} + \left(\frac{r_{j+\frac{1}{2}}}{r_{j-\frac{1}{2}}}\right)^{\nu}\right] \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} r_{j-\frac{1}{2}}^{\nu} e^{-\delta r_{j+\frac{1}{2}}} dr \\
&\le \frac{\sigma^2}{2\left(r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}\right)} \left[e^{\delta\left(r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}\right)} + \left(\frac{r_{j+\frac{1}{2}}}{r_{j-\frac{1}{2}}}\right)^{\nu}\right] \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} w_3(r) dr.
\end{aligned}$$

It follows from (4.37) that

$$\varrho h \le r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}} \le h, \quad \frac{r_{j+\frac{1}{2}}}{r_{j-\frac{1}{2}}} \le 1 + \frac{2}{\varrho}.$$

Combining the above inequalities and using $w_3(r) \le Rw_1(r)$ for $r \in (0, R]$, we get

$$\|V\|^2_{1,w_2,h} \le R\left[\frac{\sigma^2}{\varrho}\left(e^{\delta h} + \left(1 + \frac{2}{\varrho}\right)^{\nu}\right) + h^2\right] h^{-2}\|V\|^2_{0,w_1,h}.$$

This implies the desired estimate (4.43). $\square$

For $v \in C^\infty(0, R)$, define the following weighted norms

$$\|v\|^2_{0,w_1} = \int_0^R w_1(r)v^2(r)dr, \quad \|v\|^2_{1,w_2} = \int_0^R w_2(r)\left(v_r^2(r) + v^2(r)\right)dr.$$

Denoted by $L^2_{w_1}(0, R)$, $H^1_{w_2}(0, R)$ and $H^1_{0,w_2}(0, R)$ the closure of $C_0^\infty(0, R)$, $C^\infty(0, R)$ and $C_0^\infty(0, R)$ under the above norms, respectively.

Lemma 4.3. *For $V = (0, v_1, \ldots, v_{N-1}, 0)^T \in R_0^{N+1}$, let $v(r) = \sum_{j=1}^{N-1} v_j\psi_j(r)$, where $\psi_j(r)$ is defined in the same way as $\phi_j(x)$ in §4.5.2. Then there are positive constants C_1, C_2, C_3, C_4 independent of v, h and N such that*

$$C_1\|v\|_{0,w_1} \le \|V\|_{0,w_1,h} \le C_2\|v\|_{0,w_1},$$
$$C_3\|v\|_{1,w_2} \le \|V\|_{1,w_2,h} \le C_4\|v\|_{1,w_2}.$$

That is, $\|v\|_{0,w_1}$ or $\|v\|_{1,w_2}$ give an equivalent norm to $\|V\|_{0,w_1,h}$ or $\|V\|_{1,w_2,h}$ in R_0^{N+1}.

Proof. The proof is based on direct calculations as in the proof of Lemma 4.2. We omit it here. $\square$

Lemma 4.4. *There is a positive constant C_5 independent of h such that*

$$\|V\|^2_{0,w_1,h} \le C_5\|V\|^2_{1,w_2,h}, \quad \forall V \in R_0^{N+1}. \tag{4.44}$$

Proof. For $V \in R_0^{N+1}$, let $v(x) = \sum_{j=1}^{N-1} v_j\psi_j(r)$, where $\psi_j(r)$ is defined in the same way as $\phi_j(x)$ in §4.5.2. By integration by parts, we get

$$0 = \int_0^R d(w_2(r)v^2(r)) = \int_0^R w_2'(r)v^2(r)dr + 2\int_0^R w_2(r)v(r)v'(r)dr.$$

This gives

$$\int_0^R w_1(r)v^2(r)dr = \frac{1}{\nu}\left(\delta\int_0^R w_2(r)v^2(r)dr - 2\int_0^R w_2(r)v(r)v'(r)dr\right)$$

Thus, by Cauchy's inequality, we have

$$\int_0^R w_1(r)v^2(r)dr \le \frac{1+\delta}{\nu}\int_0^R w_2(r)\left(v^2(r)+v'^2(r)\right)dr,$$

i. e.,

$$\|v\|_{0,w_1}^2 \le \frac{1+\delta}{\nu}\|v\|_{1,w_2}^2.$$

Estimate (4.44) follows from the above inequality and Lemma 4.3. □

Now we can show the stability and convergence of finite volume solutions in (4.38).

Theorem 4.2. *For any $\theta \in [0,1]$, the linear complementarity problem (4.38) has a unique solution U^m for $m = 1, 2, \ldots, M$. Furthermore, if for a positive constant μ independent of h and τ,*

$$1 - 4\theta\Lambda\frac{\tau}{h^2} \ge \mu, \tag{4.45}$$

then

$$\max_{1\le m\le M}\|P^m\|_{0,w_1,h}^2 + \mu\sum_{m=1}^{M}\|P^m - P^{m-1}\|_{0,w_1,h}^2 \tag{4.46}$$
$$+\sum_{m=1}^{M}\tau\left((1-\theta)\|P^m\|_{1,w_2,h}^2 + \theta\|P^{m-1}\|_{1,w_2,h}^2\right) \le C,$$

where Λ is the positive constant in Lemma 4.2, $P^m = (g_0^m, p_1^m, \ldots, p_{N-1}^m, g_N^m)^T$, C is a positive constant independent of h, τ, but dependent on $\|G\|_{W^{1,\infty}((0,R)\times(0,T))}$ (from now on we shall use C to denote such constants). Therefore, the Backward Euler scheme ($\theta = 0$) is absolutely stable, and for $\theta \in (0,1]$ the scheme is conditionally stable.

Proof. It follows from (4.42) that A is positive definite. Thus, the linear complementarity problem (4.38) has a unique solution U^m for $m = 1, 2, \ldots, M$. In the following arguments we show the stability estimate (4.46) by proceeding along the lines of the investigation in §3.3.2 of [39].

Let $G^m = (g_0^m, g_1^m, \dots, g_N^m)^T$. Then $E^m = P^m - G^m \in R_0^{N+1}$ and from (4.41) and (4.42),

$$\|E^m\|_{0,w_1,h}^2 = (DW^m, W^m), \quad \|E^m\|_{1,w_2,h}^2 = (AW^m, W^m),$$

where $W^m = U^m - H^m$. It follows from the equality in (4.38) that

$$\left(D\delta_\tau W^m + AW^{m-\theta}, W^m\right) = \xi^m + \eta^m,$$

where

$$\xi^m = -\left(D\delta_\tau H^m, W^m\right), \quad \eta^m = -\left(AH^{m-\theta} - F^{m-\theta}, W^m\right).$$

Since

$$\begin{aligned}
(D\delta_\tau W^m, W^m) = \frac{1}{2\tau}\big[&(DW^m, W^m) - (DW^{m-1}, W^{m-1}) \\
&+(D(W^m - W^{m-1}), W^m - W^{m-1})\big], \\
\left(AW^{m-\theta}, W^m\right) = (1-\theta)&\left(AW^m, W^m\right) + \theta\left(AW^{m-1}, W^{m-1}\right) + \\
&\theta\left(AW^{m-1}, W^m - W^{m-1}\right),
\end{aligned}$$

we have

$$\begin{aligned}
(4.47) \qquad &\|E^m\|_{0,w_1,h}^2 - \|E^{m-1}\|_{0,w_1,h}^2 + \|E^m - E^{m-1}\|_{0,w_1,h}^2 + \\
&2\tau\left[(1-\theta)\|E^m\|_{1,w_2,h}^2 + \theta\|E^{m-1}\|_{1,w_2,h}^2\right] \le 2\tau(\xi^m + \eta^m + \theta\zeta^m).
\end{aligned}$$

where

$$\zeta^m = -\left(AW^{m-1}, W^m - W^{m-1}\right)$$

For any $V =\in R_0^{N+1}$, letting $\widetilde{V} = (v_1, v_2, \dots, v_N)$, we have by simple calculation

$$\left(AH^{m-\theta} - F^{m-\theta}, \widetilde{V}\right) = \sum_{j=1}^{N} b_j\left(G_j^{m-\theta} - G_{j-1}^{m-\theta}\right)(v_j - v_{j-1}) + \sum_{j=1}^{N-1} c_j G_j^{m-\theta} v_j.$$

Thus, we get by Cauchy's inequality

$$(4.48) \quad \left(AG^{m-\theta} - F^{m-\theta}, \widetilde{V}\right) \le \frac{1}{\epsilon}\|G^{m-\theta}\|_{1,w_2,h} + \epsilon\|V\|_{1,w_2,h}^2 \le \frac{C}{\epsilon} + \epsilon\|V\|_{1,w_2,h}^2$$

where ϵ is any positive number. Since

$$\begin{aligned}
\eta^m &= \eta_1^m + \eta_2^m + \eta_3^m, \\
\eta_1^m &= -(1-\theta)\left(D\delta_\tau G^m + AG^{m-\theta} - F^{m-\theta}, W^m\right), \\
\eta_2^m &= -\theta\left(D\delta_\tau G^m + AG^{m-\theta} - F^{m-\theta}, W^{m-1}\right), \\
\eta_3^m &= -\theta\left(D\delta_\tau G^m + AG^{m-\theta} - F^{m-\theta}, W^m - W^{m-1}\right).
\end{aligned}$$

by choosing proper ϵ in (4.48) for η_i^m, we obtain

$$\eta^m \le C + \frac{1-\theta}{4}\|E^m\|_{1,w_2,h}^2 + \frac{\theta}{6}\|E^{m-1}\|_{1,w_2,h}^2 + \frac{\theta}{4}\|E^m - E^{m-1}\|_{1,w_2,h}^2.$$

By using the same argument as above and Lemma 4.4, we observe

$$\xi^m \le C + \frac{1-\theta}{4}\|E^m\|_{1,w_2,h}^2 + \frac{\theta}{6}\|E^{m-1}\|_{1,w_2,h}^2 + \frac{\theta}{4}\|E^m - E^{m-1}\|_{1,w_2,h}^2.$$

By Cauchy's inequality again, we have

$$\zeta^m \le \frac{1}{6}\|E^{m-1}\|_{1,w_2,h}^2 + \frac{3}{2}\|E^m - E^{m-1}\|_{1,w_2,h}^2.$$

Substituting the above estimates in (4.47) and using Lemma 4.2 yields

$$\begin{gathered}
\|E^m\|_{0,w_1,h}^2 - \|E^{m-1}\|_{0,w_1,h}^2 + \left(1 - 4\theta\Lambda\frac{\tau}{h^2}\right)\|E^m - E^{m-1}\|_{0,w_1,h}^2 \\
+\tau\left[(1-\theta)\|E^m\|_{1,w_2,h}^2 + \theta\|E^{m-1}\|_{1,w_2,h}^2\right] \le C\tau.
\end{gathered}$$

Hence, with assumption (4.45), by summing, we get

$$\begin{aligned}
(4.49)\qquad &\max_{1\le m\le M}\|E^m\|_{0,w_1,h}^2 + \mu\sum_{m=1}^{M}\|E^m - E^{m-1}\|_{0,w_1,h}^2 \\
&+\sum_{m=1}^{M}\tau\left[(1-\theta)\|E^m\|_{1,w_2,h}^2 + \theta\|E^{m-1}\|_{1,w_2,h}^2\right] \le C.
\end{aligned}$$

This inequality leads to the desired estimate (4.46). □

Theorem 4.3. *Let $P_{h\tau}$ be the piecewise interpolation of $\{P^m\}$ linearly in r and constantly in t. Under the conditions of Theorem 4.2, $P_{h\tau}$ has a weakly convergent subsequence in $L^2(0,T;H^1_{w_2}(0,R)$.*

Proof. Let $G_{h\tau}$ be the piecewise interpolation of $\{G^m\}$ linearly in r and constantly in t. It follows from 4.49 and Lemma 4.3 that $P_{h\tau} - G_{h\tau}$ is bounded in $L^2\left(0,T;H^1_{0,w_2}(0,R)\right)$. Hence, $P_{h\tau} - G_{h\tau}$ has a weakly convergent subsequence in $L^2\left(0,T;H^1_{0,w_2}(0,R)\right)$. But it is easy to see that $G_{h\tau}$ converges to G strongly in $L^2\left(0,T;H^1_{w_2}(0,R)\right)$. Hence, $P_{h\tau}$ has a weakly convergent subsequence in $L^2\left(0,T;H^1_{w_2}(0,R)\right)$. $\square$

Since the constraint function $G(r,t)$ is neither a decreasing function of t nor in $H^2(0,R)$ for any given $t \in [0,T]$, it is hard to give a better stability estimate than (4.46). We should point out that it is unusual that the Crank-Nicholson is not absolutely stable. Since $P(r,t)$ could not possess much regularity (say $P \in W^{2,\infty}$ in r) , we can not obtain any error estimates for the finite volume methods (4.38).

Next consider the finite element method (4.40). Notice that the variational problem (4.25) and its approximation (4.40) are in the framework of §6.3.3 of [39]. We thus have the following theorems.

Theorem 4.4. *For any $\theta \in [0,1]$, the variational problem (4.40) has a unique solution w_h^m for $m = 1,2,\ldots,M$. If the conditions of Lemma 4.1 hold and for a positive constant μ independent of h and τ such that*

$$1 - \frac{16\theta\tau}{\gamma_0 \varrho h^2} \geq \mu, \tag{4.50}$$

then

$$\max_{1\le m\le M} \|w_h^m\|_{0,\Omega}^2 + \mu \sum_{m=1}^{M} \|w_h^m - w_h^{m-1}\|_{0,\Omega}^2 \\ +\gamma_0 \sum_{m=1}^{M} \tau\left((1-\theta)\|w_h^m\|_{1,\Omega}^2 + \theta\|w_h^{m-1}\|_{1,\Omega}^2\right) \le C, \tag{4.51}$$

where C is a positive constant independent of h, τ, but dependent on $\|w\|_{L^2(0,T;H_0^1(\Omega))}$.

Theorem 4.5. *Let $w_{h\tau}$ be the piecewisely constant interpolation of $w_h^m(x)$ in t. Then $w_{h\tau}$ has a weekly convergent subsequence in $L^2\left(0,T;H_0^1(\Omega)\right)$ under*

the condition (4.50). Furthermore, if

$$\theta\tau/h^2 \to 0, \quad \text{as } h,\ \tau \to 0, \tag{4.52}$$

$\{w_{h\tau}\}$ has a convergent subsequence in $L^2(0,T;H_0^1(\Omega))$.

Finally, we consider the finite element method (4.39).

Theorem 4.6. *For any $\theta \in [0,1]$, the variational problem (4.39) has a unique solution u_h^m for $m = 1, 2, \ldots, M$. If the conditions of Lemma 4.1 and (4.50) hold and $\alpha > 1/2$ or $K \le B(0,0)$, then*

$$\begin{aligned}\max_{1\le m\le M} \|u_h^m\|_{0,\Omega}^2 &+ \mu \sum_{m=1}^{M} \|u_h^m - u_h^{m-1}\|_{0,\Omega}^2 \\ &+ \gamma_0 \sum_{m=1}^{M} \tau\left((1-\theta)\|u_h^m\|_{1,\Omega}^2 + \theta\|u_h^{m-1}\|_{1,\Omega}^2\right) \le C,\end{aligned} \tag{4.53}$$

where C is a positive constant independent of h, τ, but dependent on $\|u\|_{L^2(0,T;H^1(\Omega))}$, $\|F\|_{L^\infty(0,T;H^1(\Omega))}$, and $\|F_t\|_{L^\infty((0,X)\times(0,T))}$. Therefore, $u_{h\tau}$, the piecewisel constant interpolation of $u_h^m(x)$ in t, has a convergent subsequence in $L^2(0,T;H^1(\Omega))$.

Proof. Since (4.39) can be written as a linear complementarity problem with a positive definite matrix, it has a unique solution u_h^m for $m = 1, 2, \ldots, M$. Let $F_h^m(x) = \sum_{i=0}^N F(x_i, t_m)\phi_i(x)$. Taking $v = F_h^m(x)$ in (4.39), we get

$$(\delta_\tau z_h^m, z_h^m) + a(z_h^{m-\theta}, z_h^m) \le -(\delta_\tau F_h^m, z_h^m) - a(F_h^{m-\theta}, z_h^m),$$

where $z_h^m = u_h^m - F_h^m$. Notice that $z_h^m \in H_0^1(\Omega)$ and

$$\|\delta_\tau F_h^m\|_{0,\Omega} \le R\|F_t\|_{L^\infty((0,X)\times(0,T))}, \quad \|F_h^{m-\theta}\|_{1,\Omega} \le C\|F\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)}.$$

Then (4.53) follows from the same argument in the proof of Theorem 4.2. □

In the above theorem, the condition that $\alpha > 1/2$ or $K \le B(0,0)$ assures that $F \in L^\infty(0,T;H_0^1(\Omega))$. Otherwise, $F(x,t) \notin H_0^1(\Omega)$ for any given $t \in [0,T]$. Assumptions (4.50) and (4.52) are redundant for the backward

Euler scheme, which means that the Backward Euler scheme is absolutely stable. Since $f(t, \cdot)$ is not a functional in $L^2(Q)$ for each t, stability estimates better than (4.51) could not be achieved. Also, we can not expect that $w(x,t)$ possess much regularity, for instance, $w \in H^{2,1}(Q)$. It is hard to give any error estimates for the finite element method (4.40) under the regularity that $w \in L^2(0,T; H^1(0,X))$. A similar argument applies to finite element method (4.39).

4.6 An Error-checking Method

In this section, by using a similar procedure to that introduced in [2], we shall derive an equation which only involves values of $r^*(t)$ and $P(r,t)$. By using this equation one can check the accuracy of the approximations of $r^*(t)$ and $P(r,t)$ obtained by any numerical method.

Let $\epsilon(r,t) = p(r,t) + B(r, T-t)$. It follows from (i)–(iii) in Remark 4.3 that (4.13)–(4.17) are equivalent to the following free boundary problem:

$$\epsilon_t - L\epsilon = 0, \; \epsilon > K, \quad 0 < r < r^*(T-t), \quad 0 < t \le T, \tag{4.54}$$

$$\epsilon(r^*(T-t), t) = K, \quad \epsilon_r(r^*(T-t), t) = 0, \quad 0 < t \le T, \tag{4.55}$$

$$\epsilon(r,t) = K, \quad r > r^*(T-t), \; 0 \le t \le T, \tag{4.56}$$

$$\epsilon(0,t) = Z(0,t), \quad 0 \le t \le T, \tag{4.57}$$

$$\epsilon(r,0) = Z(r,0), \quad r \ge 0, \tag{4.58}$$

where $Z(r,t) = \max(K, B(r, T-t))$. This formulation is adopted in [73]. Its advantage is that the constraint function is a constant. We may solve (4.54)–(4.58) for $\epsilon(r,t)$ by using the finite volume methods and the finite element methods given in §4.5, but numerical experiments show that the accuracy of the approximate option prices for small interest rates is poor. This may be due to the fact that $\epsilon(r,t)$ is not in the same scale as the small weight function $w_2(r)$ or the singular factor $x^{-\alpha}$ in the variable transform (4.18).

Let $Q(t) = \{(r,s) : \; 0 \le r \le r^*(T-t), \; 0 \le s \le t\}$ for $t \in [0,T]$. Integrating

equation (4.54) over $Q(t)$ and using integration by parts, we get

$$\iint_{Q(t)} \epsilon_s(r,s)drds = \iint_{Q(t)} \left(\frac{\sigma^2}{2} r\epsilon_{rr}(r,s) + (a-br)\epsilon_r(r,s) - r\epsilon(r,s) \right) drdt$$
$$= \iint_{Q(t)} (b-r)\epsilon(r,s)drds + \frac{2a-\sigma^2}{2} \int_0^t (K-\epsilon(0,s))ds - Kb \int_0^t r^*(T-s)ds,$$

where (4.55) were used. By using Green's formula,

$$\iint_{Q(t)} \epsilon_s(r,s)drds = \int_0^{r^*(T-t)} \epsilon(r,t)dr - \int_0^{r^*(T)} \epsilon(r,0)dr + K(r^*(T) - r^*(T-t)).$$

where (4.55) was used again. Combining the above equations and rearranging the resulting equality, we have

$$r^*(T-t) = r^*(T) + h(t) + b\int_0^t r^*(T-s)ds, \tag{4.59}$$

where

$$h(t) = \frac{1}{K}\Bigg(\int_0^t \int_0^{r^*(T-s)} (r-b)\epsilon(r,s)drds + \frac{\sigma^2-2a}{2}\int_0^t (K-\epsilon(0,s))ds +$$
$$\int_0^{r^*(T-t)} \epsilon(r,t)dr - \int_0^{r^*(T)} \epsilon(r,0)dr \Bigg).$$

Solving the integral equation (4.59) for $r^*(t)$, we get

$$r^*(T-t) = r^*(T)e^{bt} + h(t) + b\int_0^t e^{b(t-s)}h(s)ds.$$

By a simple calculation, $r^*(t)$ can be decomposed into two parts:

$$r^*(t) = r_1^*(t) + r_2^*(t), \quad t \in [0,T], \tag{4.60}$$

where

$$r_1^*(t) = e^{b(T-t)}\left(r^*(T) - \frac{1}{K}\int_0^{r^*(T)} \epsilon(r,0)dr \right)$$
$$+ \frac{\sigma^2-2a}{2K}\int_t^T e^{b(s-t)}(K-\epsilon(0,T-s))ds,$$
$$r_2^*(t) = \frac{1}{K}\left(\int_0^{r^*(t)} \epsilon(r,T-t)dr + \int_t^T \int_0^{r^*(s)} re^{b(s-t)}\epsilon(r,T-s)drds \right).$$

It should be pointed out that $r_1^*(t)$ is a known function of t.

An equation similar to (4.60) is employed in [2] to locate the early exercise interest rate $r^*(t)$. Here we shall use it as an error checking equation. Once an approximation $r_{old}^*(t)$ of $r^*(t)$ and $P(r,t)$ are obtained by a numerical method, we substitute their linear interpolations in the right side of (4.60) to get a new approximation $r_{new}^*(t)$ of $r^*(t)$, where the integrals can be computed by numerical integration. Now we can define the indicator of accuracy, $I(t)$ as the maximum of relative errors between $r_{old}^*(T-s)$ and $r_{new}^*(T-s)$ on $[0,t]$, i.e.,

$$I(t) = \max_{0 \le s \le t} \frac{|r_{old}^*(T-s) - r_{new}^*(T-s)|}{r_{old}^*(T-s)},$$

where t is time to the expiration. It is apparent that $I(t)$ is a increasing function of t. For a perfect method, $I(t)$ is zero and if it decreases as the grid is refined, the quality of the answer improves

4.7 Numerical Examples

In this section, we present numerical examples to examine the backward Euler scheme version and the Crank-Nicholson version of our algorithms and to compare them with the simplified binomial method in [67]. Once again our software program was written in $C++$ and run on a personal computer with a Pentium III 500 MHZ processor.

Now we outline the implementation of our methods. Recalling that the discrete problems are linear complementarity problems, one may check that the corresponding matrices satisfy the conditions of the Brennan–Schwartz algorithm ([21]). Therefore, they can be solved exactly provided that there are no rounding errors.

For the finite volume method (FVM), the approximation $p_h^m(r)$ of $p(r,t)$ at $t = t_m$ is given by

$$p_h^m(r) = \frac{r_j - r}{r_j - r_{j-1}} p_{j-1}^m + \frac{r - r_{j-1}}{r_j - r_{j-1}} p_j^m, \; r \in [r_{j-1}, r_j], \; j = 1, 2, \ldots, N.$$

For the finite element method (4.39) (FEM1), the approximation is given by

$$p_h^m(r) = \begin{cases} G(0, t_m), & r = 0, \\ r^{-\alpha/2} e^{\gamma t_m} u_h^m(\sqrt{r}), & r \in (0, R], \end{cases}$$

For the finite element method (4.40) (FEM2), the approximation is given by

$$p_h^m(r) = \begin{cases} G(0, t_m), & r = 0, \\ r^{-\alpha/2} e^{\gamma t_m} w_h^m(\sqrt{r}) + G(r, t_m), & r \in (0, R], \end{cases}$$

Then the approximation of $p(r, t)(= P(r, T - t))$, $p_{h\tau}$ is defined as

$$p_{h\tau}(r, t) = \frac{t_m - t}{\tau} p_h^{m-1}(r) + \frac{t - t_{m-1}}{\tau} p_h^m(r)$$

for $t \in [t_{m-1}, t_m]$, $r \in [0, R]$, $m = 1, 2, \ldots, M$. The approximation of $r^*(T - t)$ at $t = t_m$ for FVM is the smallest r_j such that $p_j^m = G_j^m$. The approximate free boundary determined by (4.39) or (4.40), $x_{h\tau}^*(t)$ can be computed in the same way. Hence, for FEMs, the approximate early exercise interest rate is obtained by using $r^*(t) = [x^*(t)]^2$.

In the following examples, we consider 1-year put options on a 5-year zero-coupon bond with face value \$100. The exercise price of the option is \$60. The market risk parameter is taken to be 0, that is, investors are risk neutral. From Remark 4.2, we may use σ, a and b as parameters of CIR and take $\lambda = 0$. The upper bound R for $r^*(t)$ is taken as 0.5 in all examples. Hence, the computational domain is $[0, R] \times [0, 1]$ for FVM and $[0, X] \times [0, 1]$ for FEM1 and FEM2, where $X = \sqrt{0.5}$. For simplicity we use equal distance partitions in r for finite volume methods (FVM) and in x for finite element methods. That is, $[0, R]$ and $[0, X]$ will be divided into N equal subintervals for a given positive integer N. Numerical tests show that the best approximations are obtained when the step sizes in r for FVM and in x for FEM1 and FEM2 are almost equal to the step size in time, although we can not show that the Crank-Nicholson scheme of our methods is absolutely stable. N will be determined according to this observation for a given number M of steps in time (recall that the time step size $\tau = 1/M$).

Example 4.1. In this example, we examine the convergence of the methods introduced in §3. As suggested in [67], $a/\sigma^2 < 1/4$ means a high volatility and a low speed of adjustment and $a/\sigma^2 > 1/4$ is the converse case, where $a = \kappa r_\infty$ is defined as in §4.2. For $r_\infty = 0.08$, we chose two groups of volatility σ and the speed of adjustment κ according to this discussion (see Table 4.1).

Case	σ	κ	a/σ^2	ν	δ	α	γ
A	0.5	0.1	0.032	0.064	0.8	0.032	0.0266
B	0.1	0.5	4.0	8.0	100.0	0.5078	0.2520

Table 4.1: Parameters I

Notice that the amount of computation is the same for different parameters and discrete schemes in time when $\theta < 1$. In Table 4.2, we give CPU times in seconds for the Crank-Nicholson schemes of FEM1, FEM2 and FVM in Case I.A. The CPU times include times to compute the early exercise interest rates and option prices at meshes for 1 month and 12 months to the expiration date. Actually, the approximations of $p(r,t)$, $u(x,t)$ or $v(x,t)$ was computed at all nodes of each grid. Table 4.2 shows that all of our methods are very rapid. FVM is faster than FEMs because the computation domain for FEMs is larger.

τ	1/120	1/240	1/480	1/960	1/1920	1/3840
FEM1	0.03	0.08	0.31	1.22	4.90	20.46
FEM2	0.03	0.08	0.31	1.22	4.90	20.46
FVM	0.02	0.06	0.21	0.77	3.04	11.99

Table 4.2: CPU times in seconds

Since the step sizes in r or x and in time are taken to be almost equal, we can suppose that for constants C and β independent of h and τ

$$P(r,t) - P_{h\tau}(r,t) \approx C\tau^\beta,$$

in $L^2(\Omega)$ for a given time $t \in [0, T]$. Hence we have

$$\beta \approx \beta_\tau = \frac{\log \frac{e_\tau(t)}{e_{\tau/2}(t)}}{\log 2},$$

where

$$e_\tau(t) = \frac{\left\| P_{\frac{h}{2}\frac{\tau}{2}}(r, T-t) - P_{h\tau} \right\|_{0,\Omega}}{\|P_{h\tau}\|_{0,\Omega}}$$

In Figure 4.1–Figure refe1.8 we display the log-log plots of $e_\tau(t)$ and $I(t)$ for $t = 1$ month and $t = 12$ months. We observed that that the backward Euler scheme for both the finite volume method and the finite element method has a linear convergence rate as expected. For the first group of parameters, the Crank-Nicholson scheme (CN) does not possesses a convergence rate of order 2 as expected for both methods, but it does for the second group of parameters. The reason may be that the smoothness of the option price $P(r, t)$ ($u(x, t)$ or $w(x, t)$) depends on the parameters, that is, $P(r, t)$ is quite smooth in some cases and not in other cases. We also observed that $I(t)(t = 1.0/12.0, 1.0)$ decreases as the grid is refined, much as expected, for both FVM and FEMs. This implies that our methods converge. Also, FEM1 is the most accurate method.

Next we present the differences of approximate early exercise interest rates computed by the Crank-Nicholson scheme of FEM1 and FVM or FEM2 and FVM in Figure 4.9–Figure 4.11. And the approximate early exercise interest rates computed by the Crank-Nicholson scheme of FEM1 are displayed in Figure 4.13 and Figure 4.14. All of these figures imply that our methods converge quite rapidly.

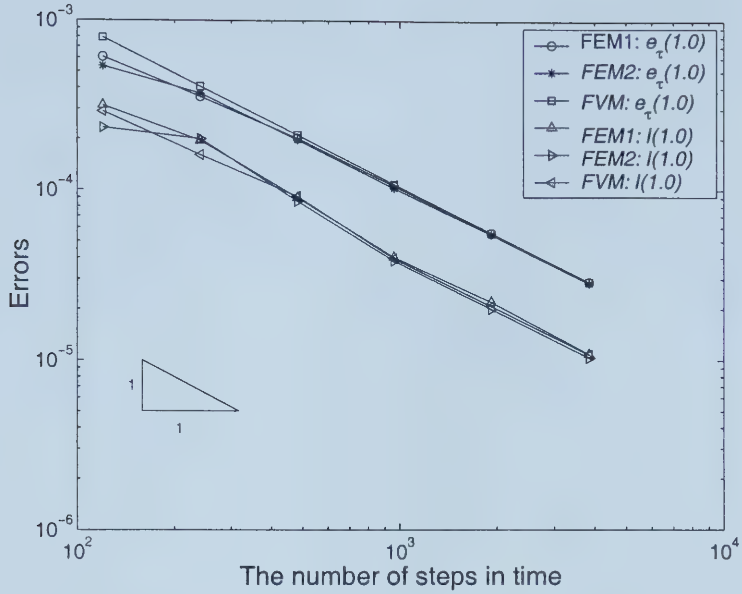

Figure 4.1: Euler scheme for Case I.A: 1 month to the expiration

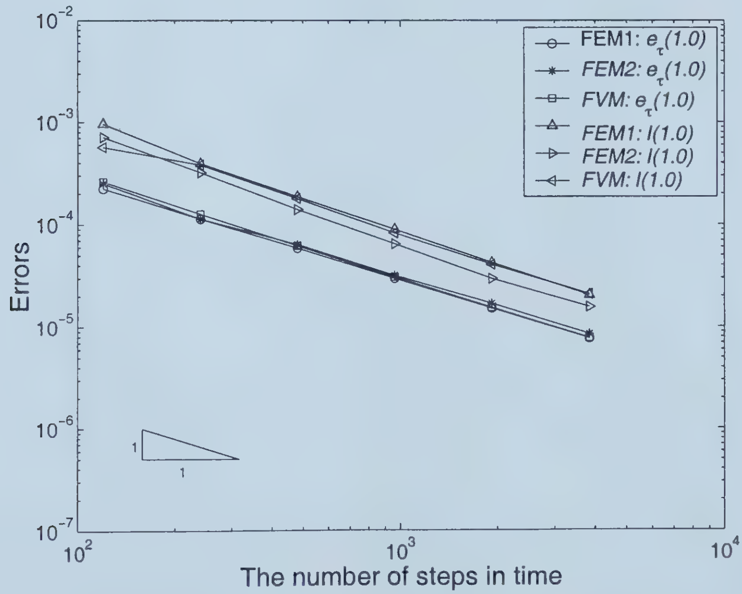

Figure 4.2: Euler scheme for Case I.A: 12 months to the expiration

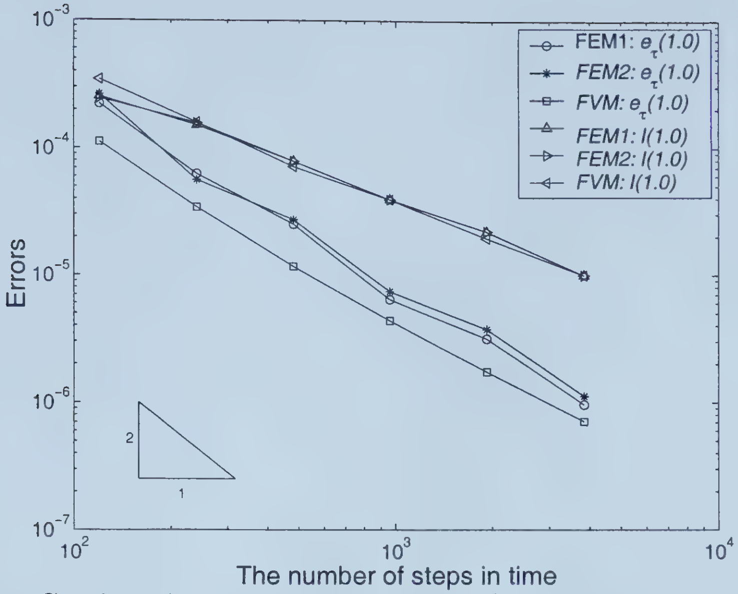

Figure 4.3: Crank-Nicholson scheme for Case I.A: 1 month to the expiration

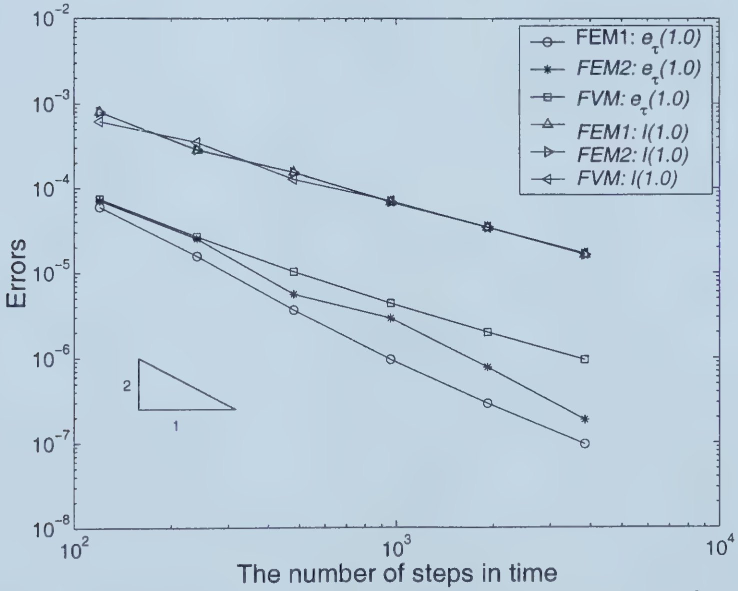

Figure 4.4: Crank-Nicholson scheme for Case I.A: 12 months to the expiration

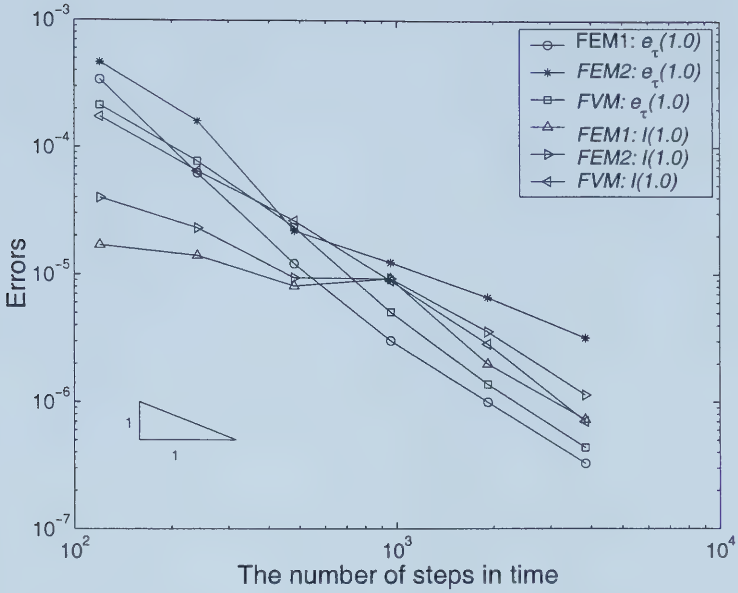

Figure 4.5: Euler scheme for Case I.B: 1 month to the expiration

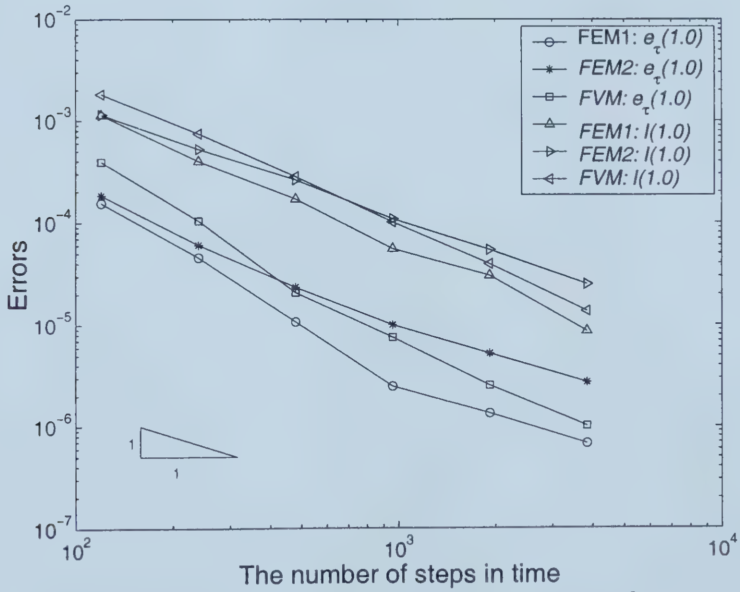

Figure 4.6: Euler scheme for Case I.B: 12 months to the expiration

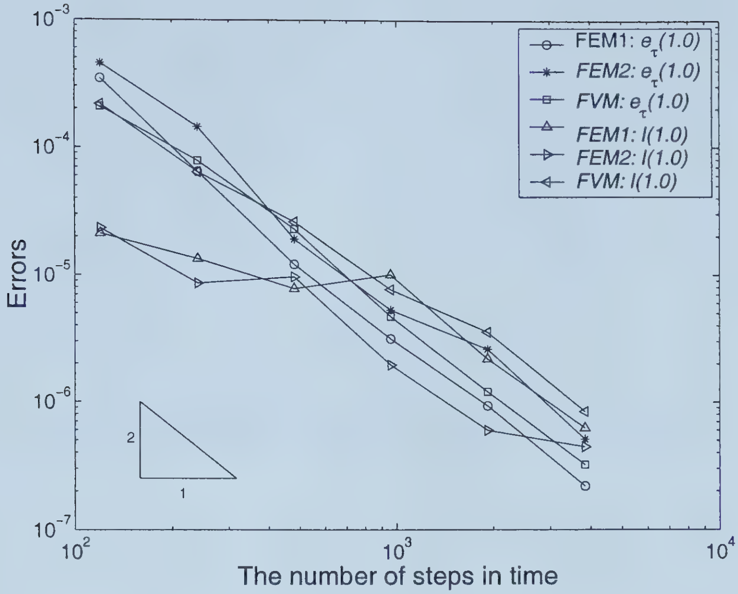

Figure 4.7: Crank-Nicholson scheme for Case I.B: 1 month to the expiration

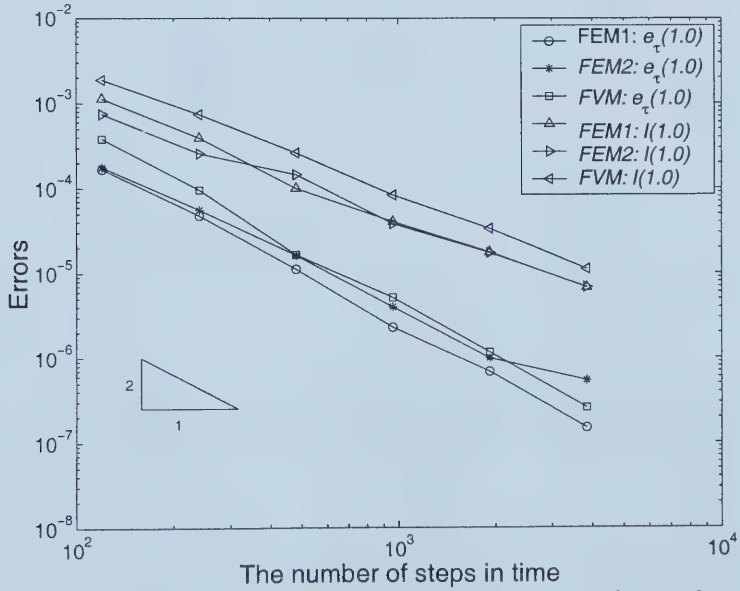

Figure 4.8: Crank-Nicholson scheme for Case I.B: 12 months to the expiration

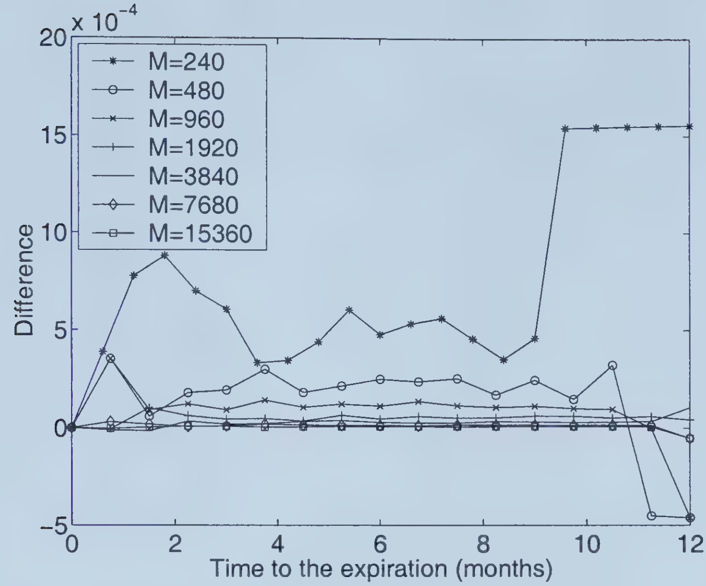

Figure 4.9: FEM1 and FVM: Case I.A

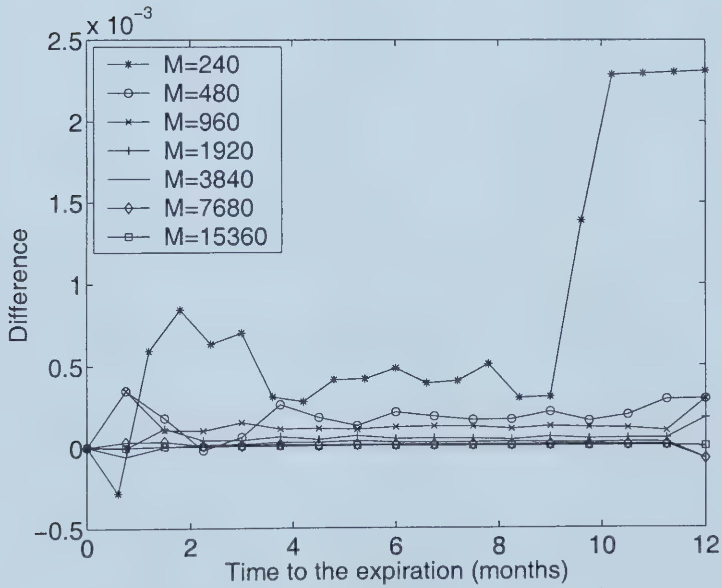

Figure 4.10: FEM2 and FVM: Case I.A

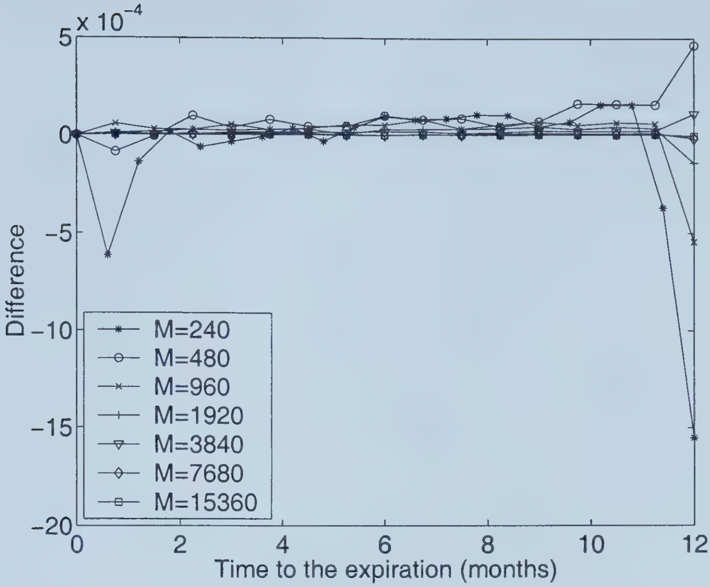

Figure 4.11: FEM1 and FVM: Case I.B

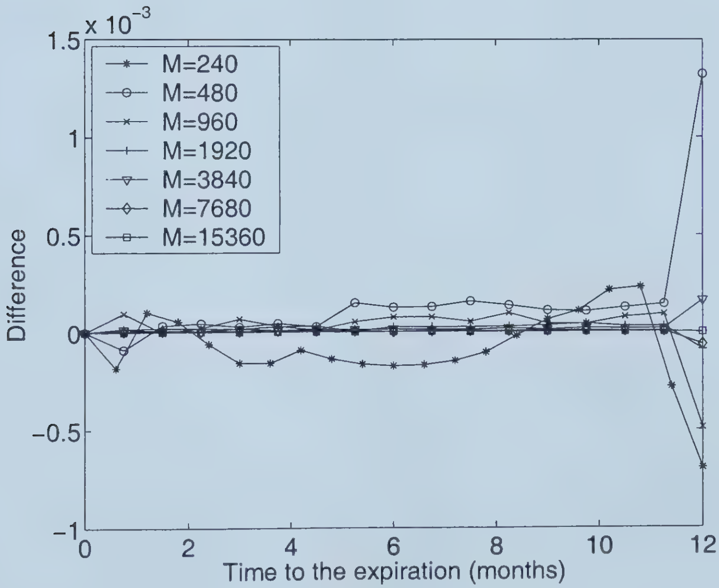

Figure 4.12: FEM2 and FVM: Case I.B

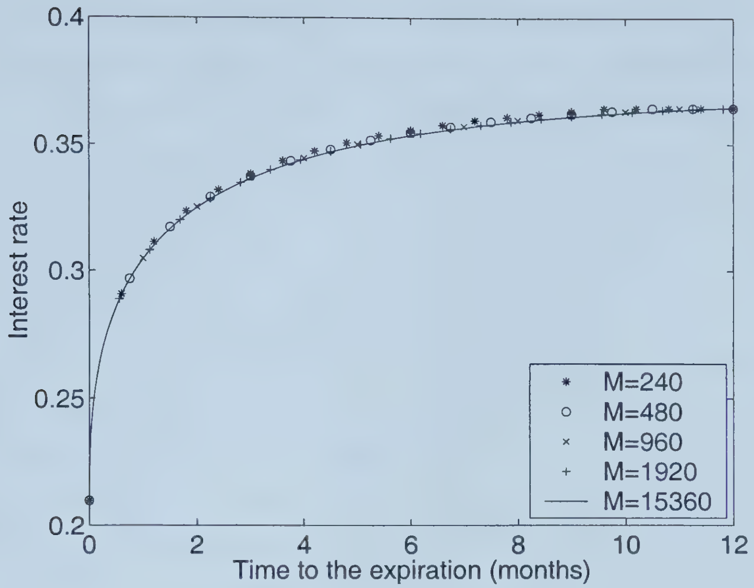

Figure 4.13: Approximations of early exercise interest rates: Case I

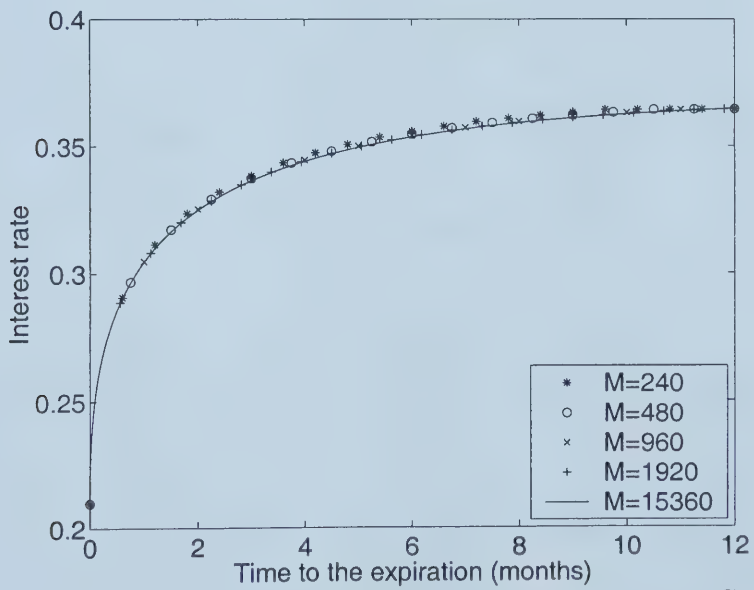

Figure 4.14: Approximations of early exercise interest rates: Case II

Example 4.2. In this example, we compare the Crank-Nicholson scheme version of FEM1 and FVM with the simplified binomial method (SB) introduced in [67]. Notice that the SB converges only when $a/\sigma^2 \geq 1/4$ as demonstrated

in [68]. The two groups of parameters in Table 4.3 satisfy this condition. We use the option price computed by the SB with 12,000 and 48,000 time steps as 'exact' for 1 month and 12 months to expiration date, respectively. In Table 4.4–Table 4.7, we display the maximum of absolute errors (MAE) at 50 points ($r = 0.01i$, $i = 0, 1, \ldots, 50$), where τ is the step size in time. We observe that FEM1 possesses the best accuracy among the three numerical methods and that SB is the poorest one, although FEM1 may not converge in a mathematically strong sense in Case II.B where $a/\sigma^2 = 1/2$. SB also converges slower than other methods. These observations should be true because SB is a kind of forward Euler finite difference method. We also see that the MAEs decrease as the time step size decreases. For the time step sizes 1/960 and 1/1920, both FEM1 and FVM achieve very good accuracy. By comparing MAEs with $I(t)$, we suggest that the approximate option price and early exercise interest rate have the desired accuracy ϵ if the step size in time can be chosen so that $I(t) = \epsilon/50$.

Case	σ	a	b	a/σ^2	ν	δ	α	γ
A	0.1	0.06	0.5	6	12	100	0.505685	0.251421
B	0.2	0.02	0.2	0.5	1.0	10	0.5	0.1

Table 4.3: Parameters II

τ	FEM1	$I(1.0/12.0)$	FVM	$I(1.0/12.0)$	SB
1/120	$6.834e{-3}$	$3.257e{-5}$	$1.523e{-2}$	$1.352e{-5}$	$3.956e{-3}$
1/480	$1.644e{-4}$	$1.239e{-5}$	$4.613e{-3}$	$2.827e{-6}$	$1.734e{-3}$
1/960	$5.680e{-5}$	$2.550e{-6}$	$8.690e{-5}$	$1.324e{-6}$	$5.881e{-4}$
1/1920	$2.395e{-5}$	$2.325e{-6}$	$8.923e{-5}$	$3.642e{-7}$	$4.524e{-4}$
1/3840	$1.393e{-5}$	$4.709e{-7}$	$6.622e{-5}$	$1.194e{-7}$	$2.312e{-4}$

Table 4.4: MAEs for Case II.A: 1 month to the expiration

τ	FEM1	$I(1.0)$	FVM	$I(1.0)$	SB
1/240	$1.342e{-3}$	$3.680e{-4}$	$1.368e{-2}$	$1.348e{-3}$	$4.617e{-3}$
1/480	$7.571e{-4}$	$2.090e{-4}$	$4.337e{-3}$	$3.410e{-4}$	$2.296e{-3}$
1/960	$7.192e{-5}$	$6.678e{-5}$	$1.133e{-3}$	$1.559e{-4}$	$1.141e{-3}$
1/1920	$5.881e{-5}$	$1.935e{-5}$	$2.517e{-4}$	$5.270e{-5}$	$5.600e{-4}$
1/3840	$2.111e{-5}$	$9.509e{-6}$	$9.021e{-5}$	$1.542e{-5}$	$2.693e{-4}$

Table 4.5: MAEs for Case II.A: 12 months to the expiration

τ	FEM1	$I(1.0/12.0)$	FVM	$I(1.0/12.0)$	SB
1/120	$3.092e{-3}$	$2.655e{-5}$	$3.110e{-3}$	$6.259e{-6}$	$8.272e{-4}$
1/480	$1.068e{-3}$	$8.865e{-6}$	$7.320e{-4}$	$1.294e{-6}$	$4.826e{-3}$
1/960	$8.286e{-5}$	$1.059e{-5}$	$2.244e{-4}$	$1.611e{-6}$	$2.583e{-3}$
1/1920	$4.186e{-5}$	$5.014e{-6}$	$4.006e{-5}$	$6.486e{-7}$	$1.660e{-3}$
1/3840	$2.012e{-5}$	$2.470e{-6}$	$3.175e{-5}$	$3.463e{-7}$	$6.040e{-4}$

Table 4.6: MAEs for Case II.B: 1 month to the expiration

τ	FEM1	$I(1.0)$	FVM	$I(1.0)$	SB
1/240	$1.254e{-3}$	$6.752e{-5}$	$3.684e{-3}$	$4.052e{-4}$	$2.908e{-3}$
1/480	$8.185e{-4}$	$7.447e{-5}$	$7.804e{-4}$	$3.336e{-5}$	$1.423e{-3}$
1/960	$1.412e{-4}$	$2.268e{-5}$	$2.032e{-4}$	$1.848e{-5}$	$7.063e{-4}$
1/1920	$6.343e{-5}$	$1.195e{-5}$	$5.776e{-5}$	$5.140e{-6}$	$3.757e{-4}$
1/3840	$4.000e{-5}$	$3.035e{-6}$	$4.519e{-5}$	$4.607e{-6}$	$3.757e{-4}$

Table 4.7: MAEs for Case II.B: 12 months to the expiration

Example 4.3. In this example we continue to examine the convergence of three methods (FEM1, FVM, SB) in Example 4.2. Since FEM1 possesses the best accuracy, we shall use the option prices obtained by FEM1 as the 'exact' solution when the step size in time is 1/60000. Parameters are specified in Table 4.8, where $\rho = a/\sigma^2 = \kappa r_\infty/\sigma^2$ as before, $r_{\max}$ is the maximum of $r^*(t)$. In Table 4.9–Table 4.18, we display the maximum of absolute errors (MAE)

at points $r = 0.005i(i = 0, 1, 2, \ldots, 100)$ for times of 1 month and 12 months to the expiration date, where τ is the step size in time. From Table 4.9–Table 4.12, we observe that SB does not converge, which does not totally agree with the conclusion in [68] that SB converges when $\rho \geq 1/4$. Examine all data in the tables, we find again that FEM1 is the best method among three methods.

Case	σ	κ	r_∞	ρ	$r_{\max}$
A	0.50	0.2	0.05	0.04	0.37992043
B	0.40	0.4	0.10	0.25	0.26584201
C	0.30	0.5	0.09	0.50	0.23744346
D	0.10	0.5	0.10	5.00	0.17074896
E	0.06	0.4	0.09	10.0	0.16833345

Table 4.8: Parameters III

τ	FEM1	$I(1.0/12.0)$	FVM	$I(1.0/12.0)$	SB
1/240	$4.653e{-3}$	$3.532e{-5}$	$2.693e{-3}$	$3.451e{-5}$	$3.529e{-2}$
1/480	$1.789e{-3}$	$2.289e{-5}$	$6.470e{-4}$	$2.629e{-5}$	$1.776e{-2}$
1/960	$1.082e{-3}$	$1.294e{-5}$	$2.516e{-4}$	$1.242e{-5}$	$9.330e{-3}$
1/1920	$3.781e{-4}$	$5.869e{-6}$	$9.783e{-5}$	$6.541e{-6}$	$4.320e{-3}$
1/3840	$3.409e{-5}$	$3.228e{-6}$	$3.954e{-5}$	$3.475e{-6}$	$2.253e{-3}$
1/7680	$1.426e{-5}$	$1.740e{-6}$	$1.647e{-5}$	$1.734e{-6}$	$1.180e{-3}$

Table 4.9: MAEs for Case III.A: 1 month to the expiration

τ	FEM1	$I(1.0)$	FVM	$I(1.0)$	SB
1/240	$1.796e{-3}$	$1.790e{-4}$	$1.742e{-3}$	$8.680e{-5}$	$7.418e{-2}$
1/480	$3.385e{-4}$	$5.828e{-5}$	$8.850e{-4}$	$5.984e{-5}$	$6.021e{-2}$
1/960	$7.117e{-5}$	$3.141e{-5}$	$4.423e{-4}$	$2.840e{-5}$	$4.924e{-2}$
1/1920	$2.340e{-5}$	$1.467e{-5}$	$2.233e{-4}$	$1.423e{-5}$	$5.797e{-2}$
1/3840	$9.888e{-6}$	$6.909e{-6}$	$1.144e{-4}$	$6.964e{-6}$	$3.449e{-2}$
1/7680	$1.049e{-5}$	$3.484e{-6}$	$5.936e{-5}$	$3.448e{-6}$	$3.692e{-2}$

Table 4.10: MAEs for Case III.A: 12 months to the expiration

τ	FEM1	$I(1.0/12.0)$	FVM	$I(1.0/12.0)$	SB
1/240	$2.427e{-3}$	$2.039e{-5}$	$1.008e{-3}$	$1.531e{-5}$	$1.905e{-2}$
1/480	$6.778e{-4}$	$9.000e{-6}$	$3.839e{-4}$	$1.117e{-5}$	$1.041e{-2}$
1/960	$1.446e{-4}$	$5.860e{-6}$	$1.703e{-4}$	$6.164e{-6}$	$5.202e{-3}$
1/1920	$5.513e{-5}$	$2.700e{-6}$	$6.000e{-5}$	$3.020e{-6}$	$2.699e{-3}$
1/3840	$4.759e{-5}$	$1.621e{-6}$	$2.494e{-5}$	$1.643e{-6}$	$1.258e{-3}$
1/7680	$2.681e{-5}$	$7.818e{-7}$	$1.059e{-5}$	$8.404e{-7}$	$6.332e{-4}$

Table 4.11: MAEs for Case III.B: 1 month to the expiration

τ	FEM1	$I(1.0)$	FVM	$I(1.0)$	SB
1/240	$3.416e{-3}$	$1.344e{-4}$	$1.393e{-2}$	$5.898e{-5}$	$2.898e{-2}$
1/480	$1.586e{-3}$	$2.146e{-5}$	$9.027e{-3}$	$2.067e{-5}$	$3.129e{-2}$
1/960	$7.619e{-4}$	$9.177e{-6}$	$5.843e{-3}$	$1.478e{-5}$	$3.257e{-2}$
1/1920	$3.708e{-4}$	$4.247e{-6}$	$3.923e{-3}$	$1.198e{-5}$	$3.163e{-2}$
1/3840	$1.794e{-4}$	$2.450e{-6}$	$2.672e{-3}$	$3.001e{-6}$	$3.266e{-2}$
1/7680	$8.510e{-5}$	$1.261e{-6}$	$1.844e{-3}$	$1.850e{-6}$	$3.279e{-2}$

Table 4.12: MAEs for Case III.B: 12 months to the expiration

τ	FEM1	$I(1.0/12.0)$	FVM	$I(1.0/12.0)$	SB
1/240	$4.309e{-3}$	$8.880e{-6}$	$1.417e{-3}$	$7.817e{-6}$	$1.232e{-2}$
1/480	$1.006e{-3}$	$7.788e{-6}$	$3.899e{-4}$	$8.860e{-6}$	$5.657e{-3}$
1/960	$2.415e{-4}$	$3.799e{-6}$	$1.609e{-4}$	$4.012e{-6}$	$2.961e{-3}$
1/1920	$4.482e{-5}$	$1.523e{-6}$	$5.015e{-5}$	$1.796e{-6}$	$1.537e{-3}$
1/3840	$1.454e{-5}$	$9.432e{-7}$	$1.799e{-5}$	$9.316e{-7}$	$6.955e{-4}$
1/7680	$5.681e{-6}$	$4.602e{-7}$	$7.920e{-6}$	$4.704e{-7}$	$3.743e{-4}$

Table 4.13: MAEs for Case III.C: 1 month to the expiration

τ	FEM1	$I(1.0)$	FVM	$I(1.0)$	SB
1/240	1.748e−3	1.046e−4	3.050e−3	5.302e−5	9.937e−3
1/480	5.381e−4	3.541e−5	2.139e−3	1.040e−5	3.791e−3
1/960	1.290e−4	1.004e−5	1.565e−3	4.803e−6	1.330e−3
1/1920	9.462e−5	2.700e−6	1.207e−3	2.260e−6	7.585e−4
1/3840	6.996e−5	1.182e−6	9.672e−4	1.174e−6	2.678e−4
1/7680	4.997e−5	8.053e−7	7.961e−4	8.691e−7	1.855e−4

Table 4.14: MAEs for Case III.C: 12 months to the expiration

τ	FEM1	$I(1.0/12.0)$	FVM	$I(1.0/12.0)$	SB
1/240	1.440e−3	2.149e−6	8.077e−3	1.295e−5	3.782e−3
1/480	9.510e−4	2.384e−6	2.228e−3	4.739e−6	1.320e−3
1/960	5.298e−4	8.379e−7	8.495e−4	1.475e−6	1.972e−4
1/1920	2.020e−4	4.676e−7	1.112e−4	5.060e−7	2.358e−4
1/3840	3.055e−5	1.399e−7	4.692e−5	1.146e−7	1.877e−4
1/7680	6.237e−6	5.099e−8	5.730e−6	7.798e−8	9.553e−5

Table 4.15: MAEs for Case III.D: 1 month to the expiration

τ	FEM1	$I(1.0)$	FVM	$I(1.0)$	SB
1/240	1.259e−2	8.050e−5	1.645e−2	1.128e−4	3.203e−3
1/480	1.883e−3	2.909e−5	2.211e−3	4.247e−5	1.692e−3
1/960	4.587e−4	1.066e−5	1.072e−3	1.325e−5	8.512e−4
1/1920	1.183e−4	4.838e−6	1.336e−4	5.009e−6	4.263e−4
1/3840	2.146e−5	1.634e−6	6.158e−5	1.628e−6	2.215e−4
1/7680	7.757e−6	6.004e−7	8.487e−6	4.465e−7	1.082e−4

Table 4.16: MAEs for Case III.D: 12 months to the expiration

τ	FEM1	$I(1.0/12.0)$	FVM	$I(1.0/12.0)$	SB
1/240	$1.592e{-2}$	$1.623e{-5}$	$4.277e{-2}$	$6.884e{-6}$	$4.515e{-3}$
1/480	$6.470e{-3}$	$2.176e{-6}$	$2.139e{-3}$	$6.501e{-6}$	$2.616e{-3}$
1/960	$2.628e{-3}$	$1.528e{-6}$	$3.605e{-3}$	$1.846e{-6}$	$6.911e{-4}$
1/1920	$4.976e{-4}$	$5.763e{-7}$	$5.152e{-4}$	$3.598e{-7}$	$2.615e{-4}$
1/3840	$5.367e{-5}$	$2.110e{-7}$	$1.778e{-4}$	$1.453e{-7}$	$3.439e{-4}$
1/7680	$1.521e{-5}$	$6.482e{-8}$	$1.271e{-5}$	$7.156e{-8}$	$5.706e{-5}$

Table 4.17: MAEs for Case III.E: 1 month to the expiration

τ	FEM1	$I(1.0)$	FVM	$I(1.0)$	SB
1/240	$6.835e{-3}$	$1.105e{-4}$	$1.733e{-2}$	$1.305e{-4}$	$3.325e{-3}$
1/480	$2.891e{-3}$	$3.683e{-5}$	$7.869e{-3}$	$5.633e{-5}$	$1.685e{-3}$
1/960	$6.571e{-4}$	$1.569e{-5}$	$1.354e{-3}$	$2.032e{-5}$	$7.637e{-4}$
1/1920	$1.138e{-4}$	$6.257e{-6}$	$4.991e{-4}$	$6.747e{-6}$	$3.917e{-4}$
1/3840	$3.978e{-5}$	$2.230e{-6}$	$8.149e{-5}$	$2.131e{-6}$	$2.034e{-4}$
1/7680	$7.320e{-6}$	$8.514e{-7}$	$3.053e{-5}$	$8.541e{-7}$	$1.018e{-4}$

Table 4.18: MAEs for Case III.D: 12 months to the expiration

Example 4.4. In this example, we shall examine the early exercise interest rate by varying the parameters: σ, κ, r_∞ and λ. Since the Crank-Nicholson scheme of FEM1 possesses the best accuracy in the above examples, we use it to compute the approximate early exercise interest rates. The number of steps in time is taken to be 2048. First we fix σ and b and vary a in Figure 4.15 and Figure 4.16, and then we fix σ and a and vary b in Figure 4.17 and figure 4.18. We observe that the early exercise interest rate is a decreasing function of a and an increasing function of b and σ. Thus it is a decreasing function of the long-term interest rate r_∞ and an increasing function of the market risk parameter λ. It is not a monotone function of the adjustment speed κ as shown in Figure 4.19 and 4.20, where we have the higher and lower volatility respectively. However, from these figures, the early exercise interest rate is an increasing function of κ near the expiration date and a decreasing

function of κ in a period at the beginning of the contract. Also $r^*(t)$ is not a monotone function of time to expiration, but rather a concave downward function. For parameters in Figure 4.16 and Figure 4.18, we have $a/\sigma^2 > 1/4$. These figures imply that the early exercise interest rate is almost a decreasing function except in a very short moment near the expiration date in the case of lower volatility. For the case of higher volatility (see Figure 4.15 and Figure 4.17), the early exercise interest rate varies very rapidly before the expiration date. Early exercise interest rates in all cases change uniformly outside a short period near the expiration date, that is, they look like straight lines.

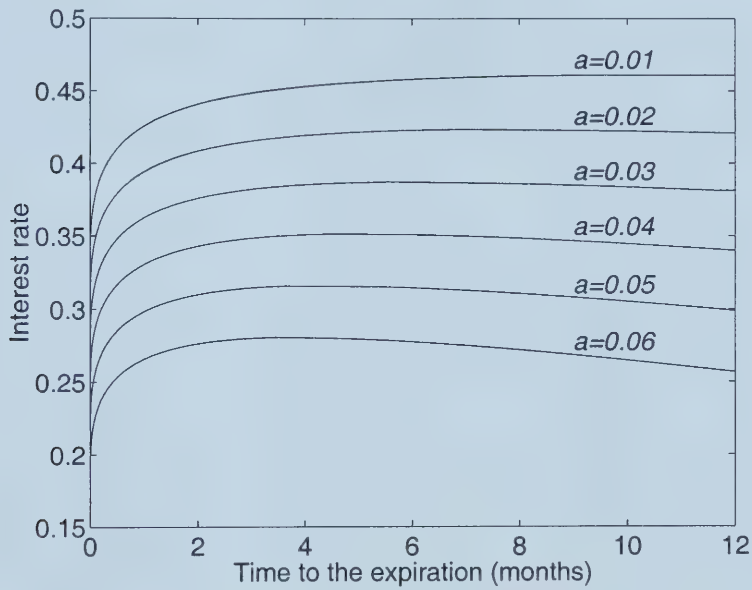

Figure 4.15: Early exercise interest rates: $\sigma = 0.5$, $b = 0.5$

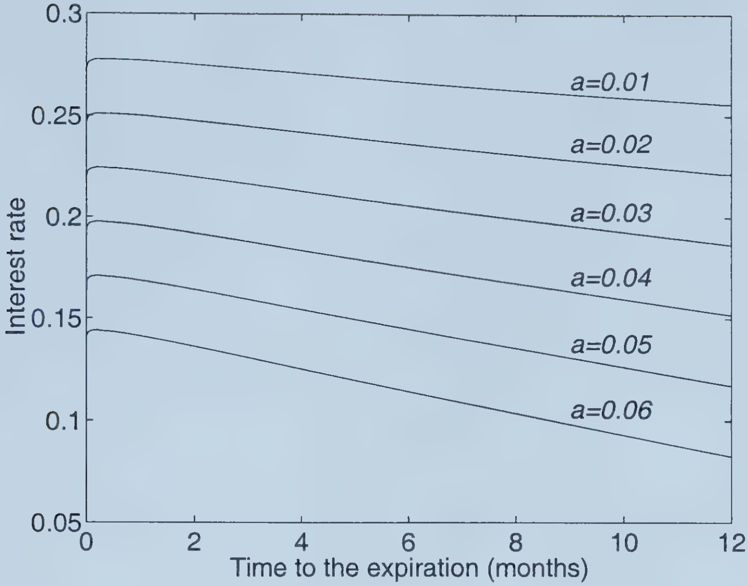

Figure 4.16: Early exercise interest rates: $\sigma = 0.1$, $b = 0.5$

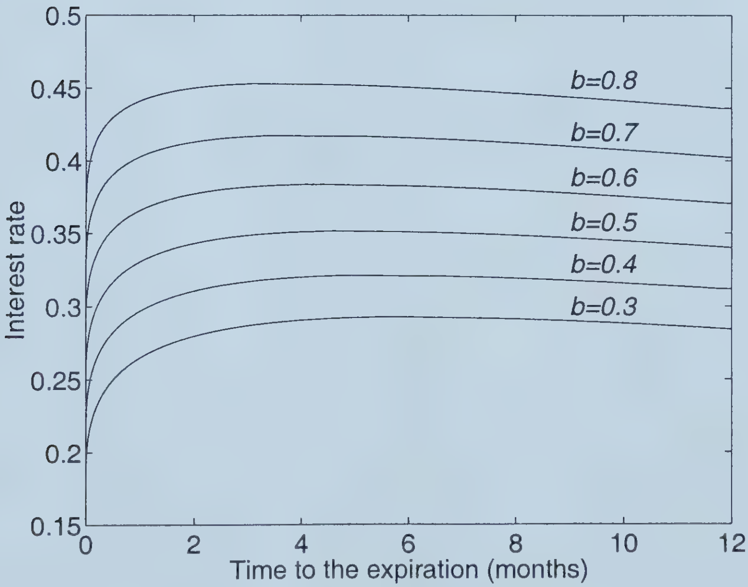

Figure 4.17: Early exercise interest rates: $\sigma = 0.5$, $a = 0.04$

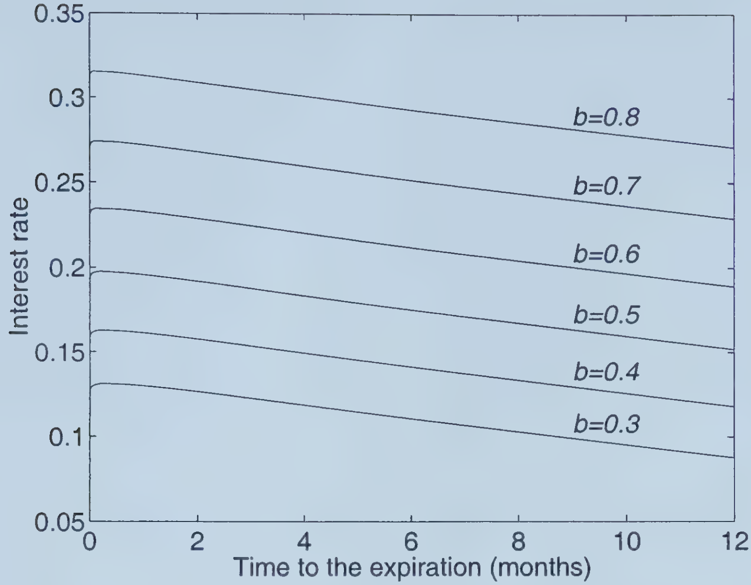

Figure 4.18: Early exercise interest rates: $\sigma = 0.1$, $a = 0.04$

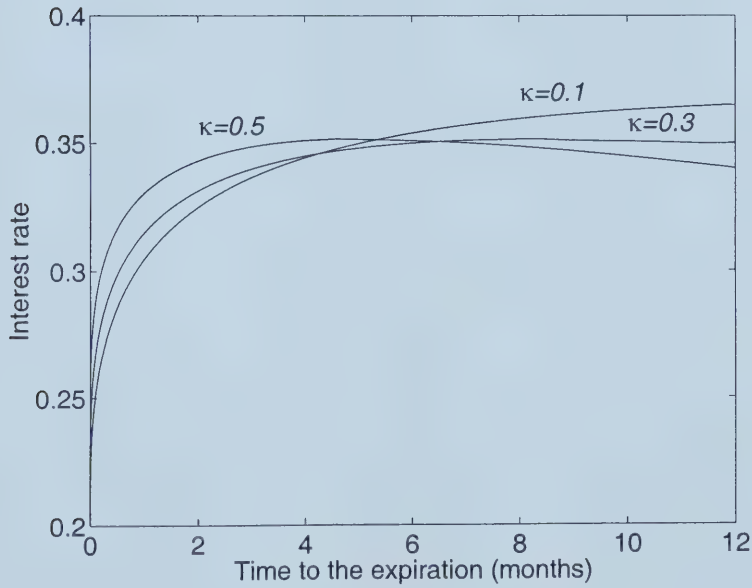

Figure 4.19: Early exercise interest rates: $\sigma = 0.5$, $r_\infty = 0.08$

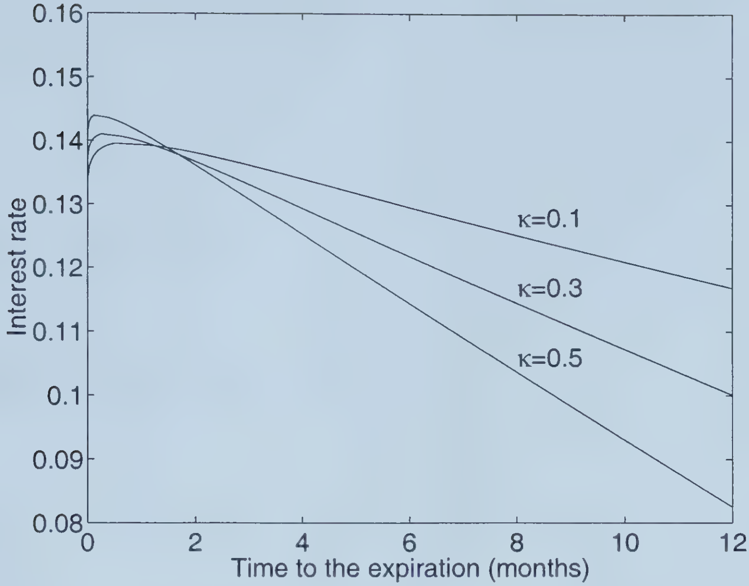

Figure 4.20: Early exercise interest rates: $\sigma = 0.1$, $r_\infty = 0.2$

Chapter 5

Conclusions

In this chapter we summarize our work in this thesis and give possible research directions.

5.1 Summary

In this thesis we have studied numerical pricing of American options on stocks and zero-coupon bonds by considering the related free boundary problems.

For American options on stocks, we reformulated the free boundary problems as variational inequalities on bounded domain by introducing exact integral-differential boundary conditions and developed a type of finite element methods. The error estimates between exact option prices and their finite element approximations under admissible regularity were established. To the best of our knowledge, there are no known error estimates for other approaches. Numerical examples show that our methods provide very rapid and accurate option prices, early exercise price, and hedge ratios. Also, our methods compare very favorably with other approaches such as the compound option approximation method, the quadratic approximation method, the binomial method, integral equation methods, and finite difference methods.

For American put options on zero-coupon bonds under the CIR model, we show the existence and uniqueness of the weak solution by formulating the cor-

responding free boundary problems as parabolic variational inequalities. This new formulation leads to a type of finite element methods which are numerically stable in a mathematically strong sense. In addition, we constructed a type of finite volume methods for the original free boundary problems. Stability and convergence are proven for the two methods. We also gave an error checking method which gauges whether or not a numerical method converges and has achieved a good accuracy. It yields a practical means of determining how fine a grid should be employed in order to obtain an answer with a desired accuracy. Numerical examples show that our methods converge in all parameter combinations and the finite element method possesses the best accuracy.

5.2 Future Work

Possible studies based on the topics of this thesis include further work on American option on bonds, for example: extending the results of Chapter 4 to the extended CIR model (see [44]) and American call options on zero-coupon bonds and numerical pricing of American options on zero-coupon bonds under other interest rate models.

Naturally, we could also consider numerical pricing of other path-dependent options, for example: binary options, compound options, chooser options, barrier options, Asian options, lookback options, Bermudan options, and so on. To the best of our knowledge, there are not many papers dealing with numerical methods for these exotic options.

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